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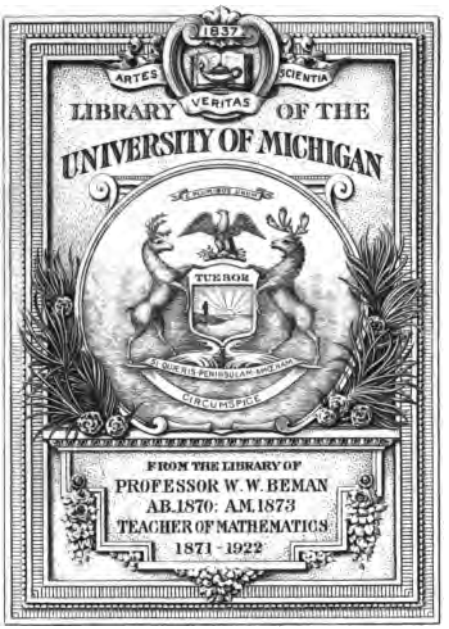
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2322. (The late Professor Townsend, F.R.S.) — Express the radius (R) of a circle orthogonal to three others in a plane, in terms of their three radii (p, q, r) and the three sides (a, b, c) of the triangle determined by their three centres. Investigate the corresponding formula for the radius of the circle orthogonal to three others on the surface of a sphere. 118

2508. (Professor Crofton, F.R.S.) — (1) A point being denoted by (ρ, σ, τ), its tri-polar coordinates, or distances from three poles R, S, T, show that all circles represented by $A\rho^2 + B\sigma^2 + C\tau^2 = 0$ are orthogonal to the circle RST. (2) If O be the centre of the circle $A\rho^2 + B\sigma^2 + C\tau^2 = D$, show that A, B, C are proportional to the triangles SOT, TOR, ROS. 62

2754. (Samuel Roberts, M.A.) — Show that, if (a, a_1), (b, b_1) are points of contact of tangents from two points on a cubic curve, and (a, b), (a_1, b_1) have the same connective, then the four points lie on a conic which passes through their tangentials. 120

2827. (Professor Sylvester.) — If four ternary quadrics in ϕ, ψ, ω, which all vanish for one and the same set of values of the variables, be respectively equal to x, y, z, t, &c.; prove, by a simple application of the dialytic method of elimination, that these last-named four quantities are connected by a cubic equation, and consequently that every cubic surface is unicursal. 57

2853. (Professor Sylvester.)—1. Let P be a given point in a cubic. Show that, in general, three conics may be drawn having contact of the 3rd order with the cubic at P and ordinary contact with it at some other point; but that no such conics (or only two) can be drawn when P is a pluperfect point of the 1st (and 2nd or 4th) order.

2. Also that, in general, twelve conics may be drawn having ordinary contact with the cubic at T, and contact of the 3rd order with it at some other point; but that only ten such conics can be drawn when T is a pluperfect point of the 2nd order. 25

2883. (Professor Sylvester.) — If there be two coincident solutions of $x+y+z=a+b+c$, $ax+by+cz=(a+b+c)^2$, $xyz=ayz+bzx+cxy$, show (1) that a curve can be formed of which a, b, c may be regarded as the trilinear coordinates, and determine its order; and find (2) how many (if any) points of the curve all three solutions will coincide; also (3) whether it is possible for the coordinates at any point of the curve to be commensurable quantities. 41

2925. (J. J. Walker, F.R.S.) — Prove that the maximum value of the angle of intersection with an ellipse at D, of the circle passing through three points on the ellipse, the osculating circles at which meet in D, is $\sin^{-1}\frac{a^2-b^2}{2(a^2+b^2)}$; and that in this case the centre of the circle lies on the chord common to the ellipse and its osculating circle at D..... 77

3302. (W. Siverly.) — Required the axes of the three greatest equal ellipses that can be drawn in an ellipse whose axes are $2a$ and $2b$..... 64

3395. (Professor Hudson, M.A.)—An indefinitely small piece of ice, the shape of which may be taken to be that of a right circular cylinder, is floating with its axis vertical in water. The part immersed receives deposits of ice in such a manner as to continue cylindrical, the radius and axis receiving equal increments in equal times. Find the ultimate shape of the part not immersed. If the specific gravity of ice be ·96, prove that the surface is formed by the revolution of $y^2(9x-y)^{25}=a^{27}$. 92

3630. (Professor Hudson, M.A.) — The centre of a spherical mass is describing a parabola freely about the focus; the sphere expands as it approaches the focus, its radius varying inversely as the square of the distance, so that it traces out a parabolic tube of variable bore. According to what power of the distance does the rate of tracing out volume vary? 120

3640. (The Rev. A. F. Torry, M.A.) — The velocity of a particle freely describing a circle varies inversely as the ordinate to a fixed diameter: find the direction and law of the force. 125

3657. (Professor Minchin, M.A.) — Show that the force by which a particle describes the inverse (with respect to centre of force) of a given curve is $\mu . r^5(r^2-2p\rho)/p^3\rho$, where r is the radius-vector to any point on the given curve, ρ and p the radius to curvature and perpendicular to a tangent at the point. 64

3707. (J. B. Saunders.)—A spherical air-bubble, having risen from a depth of 1000 feet in water, was one inch in diameter when it reached the surface; find its diameter at the bottom. 27

3729. (Professor Hudson, M.A.)—At noon, in latitude λ, the shadows upon a hill of the edges of a rectangular horizontal cloud are observed to be equally inclined at an angle γ to the horizon: regarding the hill as a plane inclined at an angle i to the horizon, with a north-westerly aspect, whose intersection with the horizon has an azimuth β east of north, find at what time of the year the observation took place. If $\lambda = 53°\,28'$, $\beta = 30° = i$, and it was at midsummer, prove that

$$\gamma = \tan^{-1}(\tfrac{1}{2}\sqrt{3}).$$ 66

3737. (J. B. Sanders.)—To find the time in which a paraboloid of revolution whose altitude is h and parameter p, full of fluid, will empty itself through a small orifice at its vertex, its axis being vertical. ... 101

3792. (G. S. Carr, M.A.)—A tube of smaller bore is joined to the open end of a common wheel-barometer tube, and alcohol is then poured in upon the mercury. Taking area of upper surface of mercury $= a$, that of the alcohol $= c$, area of common surface $= b$, sp. gr. of mercury $= m \times$ sp. gr. of alcohol; deduce the variation in the level of (c) corresponding to a variation (x) in the ordinary barometric column of mercury. 42

3799. (J. B. Saunders.) — Find the equation of the curve described by the extremities of a horizontal diameter of the air-bubble of Quest. 3707, supposing its centre to move in a vertical line. 65

3941. (For Enunciation, see Question 1423) 107

3944. (J. J. Walker, F.R.S.) — If AD is the arc drawn from one angle A of a spherical triangle so as to bisect that angle and meet the opposite side BC in D, and if the arcs bisecting the other angles meet AD in O, prove that $\frac{\tan AD}{\tan OA} = \frac{2 \sin s \cos (s-a)}{\sin (b+c)}$. 51

4054. (Rev. T. Mitcheson, B.A.) — In a spherical triangle, if $t =$ the product of the tangents of the radii of the escribed circles, show that

$$\tan r = (n^4 \sin^{-2} s \,.\, t^{-1})^{\frac{1}{2}}. \quad \text{..... 73, 103}$$

4105. (Professor Hudson, M.A.) — A wheel in the form of a cylinder of radius R and thickness A has an axle of radius r and length a cut out of the same piece, the axes and centres of gravity being coincident. The whole is suspended with the axis horizontal by three vertical strings, one of which is coiled round the wheel and the other two round the axle at equal distances on either side of the wheel: prove that, if the first string be drawn up or let down in any way, the tensions of the other two will not be altered, provided $\frac{a}{A} = \frac{R^3 (R-2r)}{r^3 (2R-r)} + 1$. 124

4794. (For Enunciation, see Question 1423) 107

7668. (Professor Sylvester.) — If m be any positive integer, and a, b, c any real quantities whatever, prove that the equation

$$a\left(1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\ldots+\frac{x^m}{1.2.3\ldots m}\right)$$
$$+b\left(1-x+\frac{x^2}{1.2}-\frac{x^3}{1.2.3}+\ldots\pm\frac{x^m}{1.2.3\ldots m}\right)+c=0$$

can never have more than two real roots. 33

7715. (W. J. C. Sharp, M.A.) — Two bags contain, the one n white balls and the other n black; and each set is numbered $1.2\ldots n$. The balls are drawn in pairs, one from each bag; find the chance, at any stage, that no pair has been drawn consisting of balls bearing the same number; and show that, as the number drawn increases, this approaches to e^{-1}.
..... 68

7850. (Asparagus.) — An equilateral triangle is inscribed in the rectangular hyperbola $x^2-y^2=a^2$; prove that the locus of the middle points of its sides is the quartic $3(x^2+y^2)^2=4(x^2-y^2)(x^2-y^2-a^2)$, or in polar coordinates $r^2=\frac{a^2\cos 2\theta}{1+2\sin 2\theta}$; and trace the curve. 30

7856. (W. J. C. Sharp, M.A.)—Construct an octahedron such that it shall have two parallel faces equal and similar to each of the faces of a given tetrahedron, and show that its volume is four times that of the tetrahedron. 30

8058. (W. J. C. Sharp, M.A.) — Prove that every unicursal twisted cubic curve is the partial intersection of three quadric surfaces, each pair of which has a common generator. 99

8457. (W. J. C. Sharp, M.A.) — If E be the spherical excess of a triangle whose sides are a, b, c, show (1) that

$\sin^2\frac{1}{4}E=\sin\frac{1}{2}s\sin\frac{1}{2}(s-a)\sin\frac{1}{2}(s-b)\sin\frac{1}{2}(s-c)/\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c$,

where $2s=a+b+c$; and hence (2) deduce the ordinary formula for the area of a plane triangle. 57

8859. (N'Importe.)—On donne dans un plan deux axes de coordonnées rectangulaires Ox, Oy, et une droite AB définie par son coefficient angulaire m et son ordonnée à l'origine b, et l'on demande : (1) De trouver la direction des diamètres des paraboles tangentes à l'axe des y au point B où il est coupé par la droite AB et ayant leurs foyers sur cette dernière droite ; (2) D'écrire l'équation générale de ces courbes ; (3) De construire le lieu de leurs sommets ; (4) De construire le lieu des points où leurs tangentes sont parallèles à l'axe des x ; (5) De construire le lieu des pôles de l'axe des x relativement aux paraboles considérés. 92

8870. (Professor Neuberg.) — Soient S le sommet d'un cône de révolution, et O le point où l'axe du cône rencontre un plan quelconque P. Démontrer que les distances de S et O à une tangente quelconque de la section du cône par le plan P sont dans un rapport constant. 43

8898. (Professor Asûtosh Mukhopâdhyây, B.A., F.R.A.S.) — Two imperfectly elastic particles, whose coefficient of elasticity is λ, are projected with equal velocities in opposite directions, at right angles to the line of force at the point of projection ; assuming the force to vary inversely as the square of the distance, and the second impact to take place midway between the point of projection and the centre of force, show that (1) $e_1=\pm(2-3\lambda^2)/\lambda^2$; (2) $e_2=\pm(3-4\lambda^2)$,

where e_1 is the eccentricity of the first orbit, and e_2 that of the orbit between the first and second impacts. Hence, determine when the eccentricity is unaltered by the impact. 34

8942. (W. J. C. Sharp, M.A.) — The polar triangles of the spherical triangles, determined by the four solid angles of a tetrahedron, exactly occupy the surface of the sphere. Construct these, and hence prove the above. 72

8995. (J. L. McKenzie, B.A., B.Sc.)—A system of coaxal circles, of which one circle S is considered fixed, is cut by a fixed line OX. If A, B be the points in which OX cuts any circle of the system, and if any line

through O cut the fixed circle S in α and β; prove that Aα, Bβ cut the circle S again in points which are collinear with a fixed point on OX. 46

9051. (J. L. Mackenzie, M.A.) — Through a point O in the plane of two conics U and V, a line is drawn cutting one conic in A and the other in B. Prove that the locus of a point P which divides AB in the ratio $\lambda : \mu$ ($\lambda + \mu = 1$) is in general a curve of the eighth order. If U and V be similar, and similarly situated, the locus reduces to a curve of the sixth order passing through the intersections of the conics $U_2 + \lambda U_1 + \lambda^2 U_0 = 0$ and $\lambda^2(U_1^2 - 4U_0U_2) - \lambda\mu V_1(U_1 + 2\lambda U_0) + \mu^2 V_0 U_2 = 0$ (where U_2 denotes terms of second degree in the equation of U, &c.), and also through the intersections of $V_2 + \mu V_1 + \mu^2 V_0 = 0$, and $\mu^2(V_1^2 - 4V_0V_2) - \lambda\mu U_1(V_1 + 2\mu V_0) + \lambda^2 U_0 V_2 = 0$. 83

9097. (W.W. Basset, M.A.)—If two tangents be drawn to the cardioid $r = a(1 - \cos\theta)$, at the ends of (α) a chord through the cusp, (β) two radii inclined at an angle $\frac{1}{3}\pi$; prove that (1) they will intersect at right angles; and (2) in the first case the locus of their point of intersection is a circle, and in the second case the locus is the limaçon $r = \frac{2}{3}a(2\cos\theta + \sqrt{3})$.... 45

9136. (N'Importe.) — Find (1) a class of curves such that the pedal with regard to a given point is the inverse with regard to that point; and show (2) that the rectangular hyperbola is a particular case. 76

9213. (D. Biddle.)—A floor is traversed by parallel equidistant lines, and a triangular plate, the diameter of whose circumcircle is equal to the distance between the lines, is thrown down at random. Given, at the circumcentre, the angles α, β, γ, which the sides of the plate subtend, find the probability that the plate lies wholly between two lines..... 116

9239. (Professor Neuberg.) — Trouver l'enveloppe d'un plan tel que les symétriques de ce plan, par rapport aux faces d'un tétraèdre donné, ABCD, concourent en un même point. 71

9294. (Belle Easton.) — PK is a chord of an ellipse normal at P, PQ, PR two chords such that the straight lines bisecting the angles between them are equally inclined to PK; prove that the locus of the point of intersection of PQ, KR is the straight line

$$\left\{(a^2+b^2)b^2 - (a^2-b^2)^2\sin^2\phi\right\}\frac{x\cos\phi}{a} + \left\{(a^2+b^2)a^2 - (a^2-b^2)^2\cos^2\phi\right\}\frac{y\sin\phi}{b} = (a^2-b^2)(b^2\cos^2\phi - a^2\sin^2\phi),$$

where ϕ is the eccentric angle of the point P. 109

9331. (A. R. Johnson, M.A.)—Present (1) as the sums of finite series,

$$\sum_{r=1}^{r=\infty} 1/\{1+(2r+1)^{2p}\},\quad \sum_{r=1}^{r=\infty} 1/\{1+(2r)^{2p}\},$$

p being any positive integer; and (2) prove that

$$\sum_{r=1}^{r=\infty} 1/(1+r^2+r^4) = -\tfrac{1}{2} + \frac{\pi}{2\sqrt{3}}\tanh(\tfrac{1}{2}\pi\sqrt{3}).$$

..... 101

9336. (Professor Morley, M.A.)—In SALMON'S *Curves*, § 216, we are told that in an acnodal cubic the acnode is the pole of the line through the three real inflexions with regard to the triangle formed by the three tangents. Here are meant the tangents at the inflexions. Prove that the proposition is also true if we take the tangents from the inflexions. 94

9480. (A. R. Johnson, M.A.)—Show that, if $O = 0$ be the condition that a pair of the lines $f(x, y) \equiv (a_0, a_1, a_2 \ldots a_n)(x, y)^n$, should be at right angles, and R denote the resultant of $f(x, y)$ and $f(-y, x)$, then

$$O = [R/\{f[1, (-1)^{\frac{1}{2}}] f[1, -(-1)^{\frac{1}{2}}]\}]^{\frac{1}{2}},$$

so that O is of order $n-1$ in the coefficients. [The axes are supposed to be rectangular.] 35

9619. (Professor Sircom, M.A.)—Find the locus of the vertices of a triangle of given species, self-conjugate with respect to a given ellipse. Discuss the various cases, and in particular examine the case in which the triangle is equilateral. 69

9719. (A. W. Panton, M.A.)—Prove that (1) there is in general one point, and only one, whose polar conic with regard to a given plane cubic is a circle, viz., the symmedian point of the triangle formed by the asymptotes; and (2) that the locus of points whose polar conics are rectangular hyperbolas is a right line, viz., the axis of perspective of the same triangle and its pedal triangle; also find (3) how the points are determined whose polar conics are—(i.) a pair of rectangular lines and (ii.) a pair of parallel lines; and (4) show that there are on the Hessian of the cubic three points forming a triangle, with reference to which any pair of corresponding points on the Hessian are isogonal conjugates. 121

9776. (W. J. C. Sharp, M.A.)—If perpendiculars p, q, r be drawn from the vertices of a triangle upon any tangent to the circumcircle, prove that these are connected by the relation

$$\begin{vmatrix} 0, & p, & q, & r \\ p, & 0, & c^2, & b^2 \\ q, & c^2, & 0, & a^2 \\ r, & b^2, & a^2, & 0 \end{vmatrix} = 0,$$

and that a similar relation holds in space of n dimensions. 55

9783. (W. J. C. Sharp, M.A.)—If ABC be a triangle, in which AC > AB, AD the perpendicular from A upon BC, and if DC′ be taken upon CD produced, so that DC′ = CD; prove that the circle through A, B, C′ will also pass through the orthocentre of ABC, and will be equal to the circumcircle of that triangle, which will pass through the orthocentre of ABC′. 44

9817. (W. J. C. Sharp, M.A.)—If A′, B′, C′ be the reflections of any point P, on the circumcircle of the triangle ABC, with respect to the sides; show, by Euclid, that A′, B′, C′ lie in a straight line which passes through the orthocentre of ABC. Hence deduce the theorem (Quest. 2145) that "the feet of the perpendiculars let fall on the sides of a triangle from any point on the circumscribing circle lie in a straight line. Show

that this straight line is equidistant from the point and from the centre of perpendiculars of the triangle." 103

9856. (J. Young, M.A. Extension of Quest. 9771.)—In a triangle the distance between the circumcentre and the orthocentre is equal to the difference of two sides $(a-b)$; prove that the angle C = 60°, or that

$$2\cos C + 3 = 4\cos(A-B).$$ 100

9952. (B. H. Steede, B.A.) — Describe a circle of a given coaxal system (of either species) cutting a given circle at a given angle. ... 123

9997. (J. C. St. Clair.)—If A, B, C, D be four points on a circle, prove that (1) the four nine-points circles of the four triangles ABC, &c. pass through the same point Q; (2) the Simson-lines of each triangle with respect to the fourth point pass through Q; (3) the four orthocentres form a quadrilateral equal to ABCD, and in perspective with it at centre Q; and (4) if a fifth point be taken on the circle, the five quadrilateral Q points lie on a circle, and so on. 113

10017. (R. Knowles, B.A.)—The circle of curvature is drawn at a point P of a conic, M is the mid-point of the common chord; the diameter of the conic through M meets the normal at P in Q; the normal meets the transverse axis in N; O is the centre of curvature at P. Prove that QN : NP = OQ : OM. 31

10031. (Professor Déprez.) — Sila droite OH qui joint le centre du cercle circonscrit au triangle ABC, à l'orthocentre, est perpendiculaire à la bissectrice de l'angle ACB, l'angle C est nécessairement égal 60°. 85

10038. (Professor Bordage.)—Show that the trilinear equation

$$\alpha(\beta-\gamma)^3 + \beta(\gamma-\alpha)^3 + \gamma(\alpha-\beta)^3 = 0,$$

represents four straight lines. 88

10046. (For Enunciation, see Question 10243; also Vol. LII., p. 83)... 88

10077. (Professor Wolstenholme, Sc.D.)—The signs of the terms of a certain series are determined by the following rule: if the first 2^r terms be taken, the signs of the second half of these terms are respectively opposite to the corresponding signs of the first half; find the sign of the n^{th} term, that of the first being positive. 53

10100. (J. W. Russell, M.A.) — It is commonly stated (without proof) in text-books that $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$ is divergent if x is positive, and greater than unity; required a proof of this statement. 124

10116. (Professor Emmerich, Ph.D.) — P, Q being the foci of an in-conic of the triangle ABC, D its point of contact with BC, prove the proportion BD : CD = b . BP . BQ : c . CP . CQ. 82

10129. (W. J. C. Sharp, M.A.)—Prove that any quadric, described through the six points in which the edges of a tetrahedron are each met by the plane through the opposite edge and a fixed point, will intersect the edges again in their six points of intersection with a system of planes through the opposite edges and another fixed point. 40

10144. (Professor Nash.) — Assuming that a knight's tour is completely determined when the first and last squares are given; show that

there are 136 distinct tours, no one of which is the reflected image of another, or can be derived from another by rotating the board. In 21 of these tours the terminal squares are a knight's move apart; if these squares be joined, show that a tour can be constructed from which each of the remaining 20 re-entrant tours can be derived either by rotation of the board or by reflection. Hence the 136 tours are reduced to 116. 36

10155. (Professor Hermite.) — Prouver que, pour toutes les valeurs réelles et positives de la variable x, l'expression $(x^{\alpha}-x^{1-\alpha})/(x-1)$, dans laquelle α désigne une quantité comprise entre zéro et l'unité, peut être représentée par $(2\alpha-1)\,\theta$, en supposant $0<\theta<1$..... 127

10168. (H. W. Segar.)—A side of a square moves with its ends on two straight lines at right angles, and touches a circle which touches those straight lines. Show that the other corners lie on two hyperbolas; find their equations referred to the fixed lines as axes. 117

10175. (Professor Truc.) — Trouver le lieu géométrique des milieux normales à une parabole donnée en supposant ces normales limitées à l'axe. 112

10185. (Professor Wolstenholme, D.Sc.) — Prove by Statical principles that, if A, B, C, D be four points on a circle, the poles of AC, BD with respect to the circle lie on the same straight line with the intersections of AB, CD, and of AD, BC. 111

10190. (Professor Morley, M.A.) — Let a line cut a cubic at ABC, let the tangential of A be a, and let the contacts of tangents from A be O, P, Q, R. Then the following conics (denoting the repetition of a point by a power) pass through the point where Ra cuts the cubic

$$R^5,\ R^3BC,\ R^3O^2,\ R^3P^2,\ R^3Q^2,\ R^2OPQ,\ RB^2C^2,\ OPQAa,$$
$$O^3PQ,\ OP^3Q,\ OPQ^3,\ OPQBC. \text{..... } 125$$

10193. (Professor Moreau.)—Trouver le lieu des centres des cercles qui passent par le sommet A d'un triangle donné et coupent les côtés AB, AC respectivement aux points B′, C′ tels que BB′ = CC′. 46

10201. (Professor Orchard, M.A., B.Sc.)—Find the relation between b and c, in order that the expression $\Sigma \equiv x^4+x^3+bx^2+cx+1=0$ may be exhibited as the product of two quadratic factors..... 39

10204. (J. C. St. Clair.) — Through each vertex of a triangle are drawn pairs of lines parallel to two fixed lines OX, OY; prove that (1) the three diagonals of the parallelograms having the sides of the triangle for their other diagonals meet in the same point P; and (2) if the sides of the constant angle XOY vary in direction, find the locus of P..... 74

10209. (H. J. Woodall. Arising out of Quest. 10048.) — If a voter may use r, $r-1$, $r-2$, ... 2, 1, 0 of his votes, show that the number of ways in which he may now act is $(n+r)\,!\,/\,(n!\times r!)$. 104

10212. (W. J. C. Sharp, M.A.)—Solve (1) the equation of differences

$$(n+1)\,P_{n-1}-(2n+1)\,P_n+nP_{n-1}=0;$$

find (2) what this solution becomes when $P_1=P_2=1$; and hence (3) show that Legendre's coefficients are each = 1, when $x=1$. 115

10214. (Rev. T. W. Openshaw, M.A.) — Prove that the sum of the squares of the perpendiculars on the tangent to an hyperbola from the foci of the conjugate hyperbola is constant. 39

10217. (M. Leroux.)—Dans un trapèze isoscèle, on connaît une base et les côtés égaux. Quelle doit être l'autre base pour que l'aire du trapèze soit maximum? 65

10223. (Professor Neuberg.) — Une droite AB de longueur constante se meut entre deux droites fixes OX, OY. On joint ses extrémités à un point fixe C du plan XOY. Déterminer géométriquement les points de contact de hauteurs AA′, BB′ du triangle ABC avec leurs enveloppes, ainsi que la tangente à la trajectoire de l'orthocentre. 116

10224. (Professor de Longchamps.)—Résoudre les équations

$$\frac{x-b-c}{x-2a}+\frac{x-c-a}{x-2b}+\frac{x-a-b}{x-2c}+1=0 \quad \ldots\ldots\ldots\ldots(1),$$

$$\frac{2x+c-b}{x+b-c}+\frac{2x+a-c}{x+c-a}+\frac{2x+b-a}{x+a-b}+2=0 \quad \ldots\ldots\ldots\ldots(2),$$

$$\frac{x+b+c-a}{x+3a-b-c}+\frac{x+c+a-b}{x+3b-c-a}+\frac{x+a+b-c}{x+3c-a-b}+1=0 \quad \ldots\ldots\ldots(3);$$

et reconnaître que la *clef* qui peut servir à former, en nombre indéfini, les équations de ce genre est représentée par l'identité

$$(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)-\alpha\beta(\alpha+\beta)-\beta\gamma(\beta+\gamma)-\gamma\alpha(\gamma+\alpha)-2\alpha\beta\gamma=0.$$

..... 51

10226. (Professor Mandart.)—On donne un cercle O, deux diamètres rectangulaires OX, OY, et une perpendiculaire CL en un point C de OX. La tangente menée au point P de la circonférence rencontre OX en A, OY en B; la perpendiculaire élevée en A sur AB coupe CL en R. Démontrer que le lieu géométrique du point d'intersection des droites OP, BR est la circonférence décrite sur OC comme diamètre. 29

10237. (D. T. Griffiths.) — The pairs of tangents to a conic from points on a straight line determine an involution on any transversal through its pole, or on any tangent to the conic. 29

10238. (J. J. Barniville.)—A triangle, in which $a^2 : b^2 : c^2 = 9 : 7 : 16$, performs a complete rotation round the side a; prove that it generates a cone whose vertical angle is equal to the trihedral angle of a cube. ... 47

10243. (Maurice d'Ocagne.) — Soit A un point arbitrairement choisi sur une hyperbole équilatère, et soient B et C les extrémités d'un diamètre quelconque de cette courbe. La tangente en A à cette hyperbole équilatère est symédiane du triangle ABC. (Ce théorème permet, connaissant le centre et deux points d'une hyperbole équilatère, de construire les tangentes en ces points.) *Corollaire.*—Si un cercle et une hyperbole équilatère sont concentriques, les tangentes à l'hyperbole aux extrémités d'un de leurs diamètres communs sont perpendiculaires à l'autre diamètre commun. 88

10244. (F. R. J. Hervey.) — Prove that the inverses of a point P with respect to four mutually orthogonal circles, invert from P and from three other points into mutually orthocentric points. 68

10247. (W. J. Greenstreet, M.A.) — A line XY is divided in Z, so that ZY = 2XZ. Draw through X a line such that chords intercepted by both circles may be equal. 128

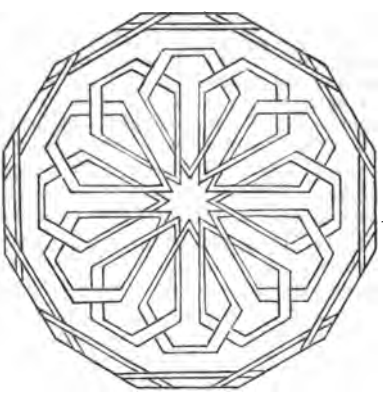
10257. (Professor Sylvester.) — If $A_1, A_2 \ldots A_i$) means the least of the quantities $A_1, A_2 \ldots A_i$, find (1) an expression for the rth in order of magnitude of the quantities $A_1, A_2 \ldots A_n$ in terms of the symbols ΣA_1; $\Sigma (A_1, A_2)$; $\Sigma (A_1, A_2, A_3) \ldots$; $\Sigma (A_1, A_2 \ldots A_{n-1})$; $(A_1, A_2 \ldots A_n)$, and show (2) the application of the question to the theory of life contingencies. 100

10263. (Professor Morley.) — Equilateral triangles have sides passing through given points; prove that the centre locus is two concentric circles, coincident when points collinear. 32

10265. (Professor Syamadas Mukhopadhyay, B.A.) — A hexagon, A, B, C, D, E, F, has three pairs of opposite sides parallel. Prove, by elementary geometry, that the triangles ACE and BDF are of equal area, and that the three lines joining middle points of opposite sides have a common point. 52

10266. (Professor Gob.) — Soient Ω et Ω' les points de Brocard du triangle ABC ($\Omega AB = \Omega' AC = \omega$). En menant par B, C, A des parallèles respectivement à ΩA, ΩB, ΩC, on forme un nouveau triangle A′B′C′. Démontrer que Ω est le point de Lemoine, et Ω' un des points de Brocard de A′B′C′. 54

10269. (F. R. J. Hervey.) — Cut out the accompanying polygon (erasing the black outline), and cut it into twelve parts, from which reconstruct the polygon, turning the whole figure into its perversion, or image in a plane mirror. [This is one of many species of symmetrical transformation obtainable by the same process, which is applicable to any regular polygon having an even number of sides; the effect produced depending upon the conditions observed in drawing the original figure.] 59



10271. (Capitaine de Rocquigny.) — Déterminer n de manière que $1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3$ soit égale à un carré. 55

10275. (D. T. Griffiths.) — The perpendiculars from the vertices on the sides of the triangle ABC meet the sides of the pedal triangle DEF at the points d, e, f. If AB and DE, AC and FD, BC and FE intersect at the points X, Y, Z respectively, prove that each side of triangle def passes through one of three points X, Y, Z. 32

10279. (Professor Orchard, M.A., B.Sc.) — Solve the simultaneous differential equations $\frac{d^2x}{dt^2} - 2y - x = 3$, $\frac{d^2y}{dt^2} - 5y - 4x = 6$.....(1, 2). 70

10283. (Emile Vigarié.) — Etant donnés un angle YOX et un point P dans son plan, on mène par ce point une droite quelconque YPX qui coupe OY en Y et OX en X. Par X et Y on trace deux droites XM, YM respectivement parallèles à deux directions données. Trouver le lieu du point M quand XY varie..... 53

10284. (J. J. Barniville.) — If C_x^y denote the combinations of y things taken x at a time, find C_x^{y+z} in terms of C_x^y and C_x^z. 49

10286. (R. Knowles, B.A.)—Two given conics osculate at a point P; their transverse axes meet the normal at P in Q and R respectively; prove that the ratio QP : RP is constant for all positions of P. 85

10294. (M. Thénard.) — Construire un triangle isoscèle ABC, connaissant un des deux côtés égaux et la somme BC + CD de la base BC et de sa projection CD sur AC. 114

10299. (Professor Neuberg.) — On projette une figure F_α, contenue dans le plan α, sur un second plan β; soit F_β la projection. On projette, ensuite, F_β sur un plan γ faisant avec β le même angle que le plan α, les traces de α et γ sur β étant perpendiculaires. Démontrer que F_γ est semblable à F_α. 38

10312. (Professor Décamps.) — Par un point P pris sur le prolongement du diamètre BA d'un cercle C, on mène une sécante quelconque PMM'; on joint le centre C aux milieux N, N' des arcs AM, AM'; les droites CN, CN' rencontrent en D et D' la perpendiculaire menée au diamètre AB par le point P. Prouver que le produit PD, PD' est constant, quelle que soit la position de la sécante. 66

10313. (Professor de Longchamps.) — Soient A un point quelconque pris dans le plan d'une ellipse E; B, C les points de contact des tangentes issues de A. (1) Démontrer que l'orthocentre du triangle ABC et le point A sont conjugués harmoniques par rapport au cercle de Monge (cercle circonscrit au rectangle des axes). Examiner le cas particulier où le point A est situé sur E, et démontrer le théorème suivant:—La normale en un point A d'une ellipse E rencontrant le cercle de Monge aux points P et Q, le centre de courbure correspondant est le symétrique, par rapport à A, du conjugué harmonique de ce point, relativement au segment PQ. 86

10315. (Professor Cesáro.)—Démontrer l'identité

$$1 - C_{2n,1}^2 + C_{2n,2}^2 + C_{2n,3}^2 + \ldots + 1 = (-1)^n C_{2n,n}. \text{ ..... } 50$$

10316. (Prof. W. S. McCay, M.A.) — Let P, Q be the feet of the perpendiculars from the incentre of a triangle on the perpendicular bisector of the base, and from the circumcentre on the bisector of the vertical angle. Prove that PQ passes through the incentre of the triangle formed by the mid-points of the sides of the given triangle. 71

10318. (Professor Guichard.)—Etant donné un triangle ABC, détermine dans son plan un point P tel qu'en menant la sécante PA'B' qui coupe AC en A' et BC en B', le point d'intersection M de AB', A'B décrive un cercle quand la sécante tourne autour du point P. 40

10324. (R. H. W. Whapham, B.A.)—If a parabola be drawn having contact of the third order with a conic, prove that the axis of the parabola is parallel to the line joining the point of contact to the centre of the conic. 50

10325. (D. T. Griffiths.) — O is the circumcentre of the triangle ABC, and P a variable point in AB (or in AB produced). The circle on OP as diameter meets AC in D, and PD is produced to E, so that DE = PD. Show that the locus of intersection of BD, CE is the circumcircle ABC. 53

10327. (E. M. Langley, M.A.)—Deduce geometrically the existence of the symmedian point from that of the Brocard-points. 99

10329. (Prof. Orchard, M.A., B.Sc.) — Show that the roots of the equation

$$x^6+(x^2-x)^4+(x^2-2x)^4+(x^2-3x+2)^4+9(x-1)^4+7(x-2)^4+16x^4+63=0$$

are $1\pm(-3\pm 7^{\frac{1}{2}})^{\frac{1}{2}}$, $\frac{1}{2}\{1\pm[-3\pm(-3)]^{\frac{1}{2}}\}$. 64

10332. (R. Knowles, B.A.) — Prove that (1) the trilinear coordinates of the centroid of the pedal triangle are proportional to

$$a^2\cos(B-C),\quad b^2\cos(A-C),\quad c^2\cos(A-B);$$

and thence (2) that $a^3\cos(B-C)+b^3\cos(A-C)+c^3\cos(A-B)$

$$=a(b^2+c^2)\cos A+b(a^2+c^2)\cos B+c(a^2+b^2)\cos C$$ 104

10336. (Maurice d'Ocagne.) — Les perpendiculaires élevées par le sommet C et par le milieu M du côté BC, à ce côté, coupent le côté AB respectivement en D et en P. La perpendiculaire élevée en C à CP coupe PM en Q. Démontrer que la droite DQ et la perpendiculaire élevée en C, à CA, se coupent sur la symédiane issue de A du triangle ABC. ... 79

10337. (H. W. Segar.)—A fixed point P is taken on the auxiliary circle of a rectangular hyperbola, and Q, R are the ends of any diameter of the conic. O is the centre of the circle PQR; O has a certain locus. Find, whatever be the position of P, the equations to the tangents at the centre of the conic to these loci. 70

10338. (C. Bickerdike.)—If the density at any point of a triangular lamina varies as the product of the perpendiculars on the sides, prove that the mean density is $\frac{8}{\pi^2}$ of the density at the centres of inertia of the area of the triangle..... .. 49

10339. (L. Bénézech.) — On donne deux cercles O, O' et un point P. On mène deux rayons parallèles OA, O'A'; sur PA, PA' comme côtés on construit un parallélogramme. Pour quelle position des rayons parallèles la diagonale issue de P est-elle égale à la somme des rayons des cercles O, O'?..... 128

10345. (Professor de Longchamps.) — Une ellipse étant rapportée à ses axes, démontrer que les coordonnées d'un point quelconque de la

normale à l'ellipse, en un point dont les coordonnées sont $a\cos\phi$, $b\sin\phi$, peuvent être représentées (λ étant paramètre variable) par les formules

$$x = \frac{c^2\cos\phi}{a}\cdot\frac{\sin\lambda\sin\frac{1}{2}(\lambda+\phi)}{\cos\frac{1}{2}(\lambda-\phi)},\quad y = -\frac{c^2\sin\phi}{a}\cdot\frac{\cos\lambda\cos\frac{1}{2}(\lambda+\phi)}{\cos\frac{1}{2}(\lambda-\phi)}.$$

..... 35

10346. (Professor Neuberg.) — Etant données deux ellipses concentriques et homothétiques, on mène en un point quelconque M de l'ellipse intérieure la normale qui rencontre les axes en P, Q; la tangente en M rencontre l'ellipse extérieure en R, S. Démontrer que les angles RPS, RQS sont constants..... 67

10350. (Professor Sollertinski.) — Une circonférence Δ, passant par le centre O d'une autre circonférence Δ', coupe celle-ci en A, B. Une corde AM de Δ' rencontre Δ en M'. Démontrer (1) que BM' = MM'; et (2) déduire une solution du problème suivant :—Deux points A, B, étant donnés sur une circonférence Δ, trouver sur Δ un point M' tel que M'A + M'B soit égale à une longueur donnée $2h$. 41

10352. (Professor Curtis.)—The conics

$$ax^2+2hxy+by^2+2gx+2fy+c=0,\quad a'x^2+2h'xy+\ldots+c'=0,$$

cut each other in four points; prove that ordinates of these points are given by $Py^4+Qy^3+Ry^2+Sy+T=0$, where

$P = 4(ah')(hb') - (ab')^2$,

$Q = 4\left[(ag')(hb') + (ah')\left\{(gb')+2(hf')\right\} - (ab')(af')\right]$,

$R = 4\left[(ag')\left\{2(hf')+(gb')\right\} + (ah')\left\{(hc')+2(gf')\right\} - (af')^2 - \frac{1}{2}(ab')(ac')\right]$,

$S = 4\left[(ah')(gc') + (ag')\left\{(hc')+2(gf')\right\} - (af')(ac')\right]$,

$T = 4(ag')(gc') - (ac')^2$..... 119

10353. (The Editor.) — Show that, if $s = \frac{1}{2}(a+b+c)$, $s_1 = s-a$, &c., the solution of the equations

$$\frac{x+y}{a} = \frac{y+z}{b} = \frac{z+x}{c},\quad mxy+nyz+pzx = xyz$$

is given by $$z = s_1\left(\frac{m}{s_1}+\frac{n}{s_2}+\frac{p}{s_3}\right),\ \text{&c.}$$ 52

10354. (G. F. Howse, M.A.)—Evaluate the determinant

$$\begin{vmatrix} c & a & a & a & a & \ldots \\ b & c & a & a & a & \ldots \\ b & b & c & a & a & \ldots \\ b & b & b & c & a & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \end{vmatrix}$$

of the nth order, all the terms in the diagonal being c, those to the right being a, those to the left b. 38

10356. (Benjamin Williamson, F.R.S.)—Given an elementary proof of Stokes' theorem, viz.—If a portion of a surface S be bounded by a closed curve s, and if u, v, w be continuous functions of x, y, z, then

$$\left\{\iint l\left(\frac{dw}{dy}-\frac{dv}{dz}\right)+m\left(\frac{du}{dz}-\frac{dw}{dx}\right)+n\left(\frac{dv}{dw}-\frac{du}{dy}\right)\right\}dS = \int(udx+vdy+wdz),$$

where l, m, n are the direction cosines of the normal to the surface at the

point xyz, and the double integral is taken over the surface, and the single integral along its boundary in the positive direction. 34

10361. (Maurice d'Ocagne.) — Soit M un point pris sur la base BC du triangle ABC. Si la perpendiculaire élevée en A à la droite AM coupe en B′ et en C′ les perpendiculaires élevées respectivement à AB et à AC, en B et en C; démontrer que (1) $AB'.MB/AC'.MC = AB^2/AC^2$; et (2) si AM est symédiane de ABC, le sommet A est le milieu de B′C′..... 49

10372. (J. J. Barniville.) — If the normal at any point of a conic divide the abscissa in the ratio $1 : [1-(1-\&c.)^{\frac{1}{2}}]^{\frac{1}{2}}$, prove that the area of the curve is equal to that of a concentric circle through the foci. 72

10373. (W. J. Greenstreet, M.A.) — Through C, the mid-point of the arc ACB of a circle, any chord CE cuts the straight line AB in D; find the locus of the centre of the circle through B, D, E. 47

10374. (R. Tucker, M.A.)—The "N.P." circle of a triangle is the auxiliary circle of the in-ellipse $(a^2\cos A\alpha)^{\frac{1}{2}} + (b^2\cos B\beta)^{\frac{1}{2}} + (c^2\cos C\gamma)^{\frac{1}{2}} = 0$, of which the foci are the circum- and ortho-centres. If L, M, N are the points of contact with the sides, prove that AL, BM, CN meet in the Isotomic of the circumcentre. 95

10378. (Professor Wolstenholme, M.A., Sc.D.)—From a point O are drawn straight lines OP, OP′ touching a given parabola in P, P′; find (1) the locus of O when the ratio OP : OP′ is given; also prove that (2) this locus cuts the directrix at right angles in two points; (3) if from either of these points O the tangents OP, OP′ meet the latus rectum in L, L′, and LQ, L′Q′ be also tangents in Q, Q′ to the parabola,

$$OP : OP' = LP : LQ = L'Q' : L'P';$$

and (4) the focal distance of any point on the locus will be the least possible when the point lies on the tangent at the vertex of the parabola (being the point of contact of a common tangent to the locus and to the parabola), and this minimum distance bears to the distance between focus and vertex the given ratio of the tangents. 91

10385. (Professor Gob.)—Soit ABCD un parallélogramme quelconque, et soient A′, B′, C′ les seconds points de rencontre des droits DA, DB, DC avec la circonférence circonscrite au triangle ABC. Démontrer que la droite A′D est la moyenne géométrique, en grandeur et en position, des droites A′C′ et A′B′. 48

10387. (The Editor.)—Solve the equations

$$x^2(y-z) = a, \quad y^2(z-x) = b, \quad z^2(x-y) = c. \ldots\ldots 86$$

10393. (Rev. G. Richardson.) — If the transversals AD, BE, CF of the triangle ABC divide the opposite sides in the ratios $l : 1$, $m : 1$, $n : 1$, respectively, prove that (1) the area of the triangle they form : area ABC

$$= (lmn-1)^2 : (mn+m+1)(nl+n+1)(lm+l+1);$$

(2) the two triangles are similar if

$$(l+1)(mn+m+1), \ (m+1)(nl+n+1), \ (n+1)(lm+l+1)$$

are in the duplicate ratios of the sides of ABC; and (3) if any one of the transversals is trisected by the other two, $4lmn = 1$. 87

10398. (Capitaine de Rocquigny.) — En un point quelconque M d'un quadrant de cercle AB, on mène une tangent, rencontrant le rayon OA en N. On élève sur OA la perpendiculaire NT = NM, et l'on joint AT, MT. Démontrer que (1) MT passe par un point fixe; (2) le lieu de T est une hyperbole équilatère (théorème connu); (3) le point d'intersection des droites NM et AT se meut sur une hyperbole. 83

10400. (D. Biddle.) — Three equal spheres touch each other. Find the probability that the plane which cuts one cuts all..... 80

10402. (E. M. Langley, M.A.) — If a diameter of one of two orthogonal circles cuts the common chord at I, and the other circle in C, and if AB is any chord of the first circle through I, show that CI is a symmedian of the triangle ABC. 60

10410. (Professor Curtis, M.A.) — Prove that there holds among any four points on an equilateral hyperbola the relation (analogous to Ptolemy's theorem) that if the points be 1, 2, 3, 4,

$$(12 \cdot 34)^{\frac{1}{2}} \pm (23 \cdot 14)^{\frac{1}{2}} \pm (31 \cdot 24)^{\frac{1}{2}} = 0,$$

where 12 = the area of a right-angled triangle whose base is the line joining 1 and 2, and the sides of which are parallel to the asymptotes of the hyperbola. 63

10411. (Professor Wolstenholme, M.A., Sc.D.) — On a side BC of a given triangle ABC are described two equilateral triangles aBC, a'BC; a, A being on the same side of BC; prove that Aa is three times the difference, Aa' three times the sum of the semi-axes of the maximum ellipse inscribed in the triangle ABC, and that the axes of this ellipse are parallel to the bisectors of the angles between Aa, Aa'. 73

10412. (Professor Neuberg.) — Etant donnés un tétraèdre ABCD et un point P, on mène par P un plan coupant l'angle solide A suivant un triangle T_1, ayant pour centre de gravité P. Soient, de même, T_2, T_3, T_4 des triangles ayant pour centre de gravité P et leurs sommets sur les arêtes des trièdres B, C, D du tétraèdre. Démontrer que les quatre triangles T_1, T_2, T_3, T_4 sont inscrits à une même surface du second ordre. 74

10414. (Professor de Longchamps.) — Soient OX, OY deux rayons rectangulaires dans un cercle Δ. Par O, on trace deux rayons OA, OB, symétriques par rapport à la bissectrice de YOX. Par B, on mène des parallèles aux droites OX, OY; elles rencontrent OA aux points I, J. Démontrer que la polaire de I, relativement à Δ, passe par J. 81

10425. (Professor Tarry.) — Quatre trains se meuvent sur des voies rectilignes avec des vitesses uniformes. On donne leurs positions à deux instants différents. Construire une voie rectiligne qui puisse être parcourue par un train avec un vitesse uniforme telle que les quatre premiers trains paraissent immobiles au conducteur du cinquième. 105

10429. (A. E. Jolliffe.) — A parabola touches the sides of a triangle at A', B', C'; prove that AA', BB', CC' meet on the minimum ellipse circumscribing the triangle. 60

10431. (W. J. Greenstreet, M.A.) — A spherical raindrop, radius a, falls for t'', when its radius is found to be b. Supposing the increase of mass to be proportional to the surface, find its velocity at the end of the t''. 119

10432. (R. Knowles.) — Tangents TP, TQ are drawn to meet a conic in P and Q; a second conic, centre O, through TPQ, cuts the first again in C, D; T′ is the pole of CD with respect to the first conic; the chords PQ, CD meet in E; prove that EO bisects TT′. 61

10433. (H. J. Read, M.A.)—Trace the curve

$$(x+y)(x-y)^2-4(x^2+y^2)+5y=0. \quad \text{..... } 61$$

10435. (Maurice d'Ocagne.)—Soient F le point où la symédiane issue de A du triangle ABC coupe le cercle circonscrit à ce triangle, $F\beta$ et $F\gamma$ les perpendiculaires abaissées de ce point sur AB et sur AC. Si par les sommets B et C on mène deux antiparallèles quelconques BB′ et CC′ relativement à l'angle BAC, et que ces droites coupent respectivement $F\beta$ et $F\gamma$ en B_1 et en C_1, la parallèle à AB menée par B_1 et la parallèle à AC menée par C_1 se coupent sur la tangente en A du cercle ABC. 98

10440. (Professor Haughton, F.R.S.) — Describe the quartic curve passing through the following points, whose Cartesian coordinates (with $x = 104$ as origin) are (1) Rubidium (85, −1); (2) Strontium (87, −2); (3) Yttrium (89, −3); (4) Zirconium (90, −4); (5) Niobium (94, −3); (6) Molybdenum (96, −2); (7) Hecaton (100, −1); (8) Silver (108, 1); (9) Cadmium (112, 2); (10) Indium (113, 3); (11) Tin (118, 4); (12) Antimony (121, 3); (13) Tellurium (125, 2); (14) Iodine (127, 1). 89

10441. (Prof. Orchard, M.A.) — Give (1) an algebraical proof of the following result:—Having put down any sum of money (£. *s*. *d*.): (*a*) Reverse (so that pence come under pounds), and subtract; (*b*) Reverse this last sum, and add; then the result is constant, and is £12. 18*s*. 11*d*., as shown in the annexed example. Also (2) generalise the theorem.

£	*s*.	*d*.
10	17	5
5	17	10
4	19	7
7	19	4
12	18	11

..... 78

10444. (Professor Curtis, M.A.)—Prove that the equation of an equilateral hyperbola touching three equilateral hyperbolas (the four curves being similarly placed) is

$$(23 \cdot S_1)^{\frac{1}{2}} \pm (31 \cdot S_2)^{\frac{1}{2}} \pm (12 \cdot S_3)^{\frac{1}{2}} = 0,$$

where 23 is the area of a triangle of which the base is a common tangent to S_2 and S_3, and the sides are parallel to the asymptotes. 63

10448. (Professor Neuberg.) — Deux droites *d*, *d*′ roulent sur deux courbes données Δ, Δ′ avec des vitesses angulaires, qui sont constamment égales et de sens contraire. Trouver la tangente à la courbe décrite par le point d'intersection de ces droites. 90

10451. (Professor Buenger.) — Démontrer qu'il existe deux points dont les distances aux sommets d'un tétraèdre donné ABCD soient proportionnelles à des nombres donnés α, β, γ, δ. Ces points divisent harmoniquement un diamètre de la sphère circonscrite au tétraèdre ABCD. 97

10453. (Professor Gob.) — Soient F_a, F'_a les foyers d'une hyperbole tangente au côté BC d'un triangle ABC et asymptote aux côtés AB, AC.

Soient F_b, F_b' et F_c, F_c' les foyers des deux autres hyperboles analogues. Démontrer que les quadrilatères $BF_a\,CF_a$, $F_b\,F_c\,F_b'\,F_c'$ sont harmoniques. 80

10456. (The Editor.) — In the base BC of a triangle find a point P, such that $AP^2 = m \,.\, BP \,.\, CP$. 81

10461. (A. E. Jolliffe.) — A conic has a given self-conjugate triangle, and its director circle passes through a fixed point. Construct four fixed tangents to it. If a conic be drawn to circumscribe the triangle with the fixed point as focus, the corresponding directrix touches the given conic. 120

10462. (W. J. Johnston, M.A.) — If a vector $\equiv ix + jy + kz$, as in quaternions, and we assume

$$i^2 = j^2 = k^2 = ijk = +1 \quad \text{(so that } ij = ji = k, \text{ \&c.)};$$

prove that (1) an *associative* and *commutative* algebra is thereby obtained; (2) on this system the scalar part of

$$a\,(b-c) + b\,(c-a) + c\,(a-b) = 0,$$

gives the known theorem that, if A, B, C, D are four points in space,

$$\Sigma\, AB \,.\, CD \cos (\overset{\frown}{AB, CD}) \equiv 0;$$

(3) interpret the vector part of the product of two vectors, also the vector part of the above identity; and (4) if $w \equiv i + j + k$, prove $(w+1)(w-3) = 0$, and interpret this. 121

10467. (Walter Stott.)—Construct apentagon when there are given the mid-points of (1) the diagonals or (2) the sides. 82

10468. (Maurice d'Ocagne.) — Soit μ la droite harmoniquement associée au point M, par rapport au triangle ABC. Les droites joignant un point quelconque de μ à A, à B, et à C, coupent les pédales correspondantes du point M en α, β, et γ. Démontrer que les côtés du triangle $\alpha\beta\gamma$ passent par les sommets du triangle ABC.

[La droite harmoniquement associée au point M par rapport au triangle ABC est celle qui joint les conjugués harmoniques, par rapport aux extrémités des côtés de ce triangle, des points A', B', C', où ces côtés sont respectivement coupés par AM, BM, CM. Les pédales du point M sont les droites A'B', B'C', C'A'.] 102

10474. (J. Moreau.)—De l'un des sommets d'un triangle, on mène des tangentes à un cercle quelconque passant par les deux autres sommets. Trouver le lieu du point où la corde de contact rencontre le diamètre qui passe par le premier sommet. 96

10475. (Professor Poulain.) — Dans un triangle, démontrer que la distance d du centre de gravité au point de Lemoine vérifie l'égalité

$$9d^2\,(\Sigma a^2)^2 = -\Sigma a^6 + 3\Sigma a^4 b^2 - 15 a^2 b^2 c^2. \quad \ldots\ldots 97$$

10476. (Professor Sylvester.)—Prove (1) that it is always possible to transform $ax\xi + bx\eta + cx\zeta + a'y\xi + b'y\eta + c'y\zeta + a''z\xi + b''z\eta + c''z\zeta$, where all the coefficients are real, into the form $Ax\xi + By\eta + Cz\zeta$, by means of two *real* orthogonal linear substitutions imposed on x, y, z and ξ, η, respectively; and show (2) how to construct the equation for determining A, B, C. 90

10479. (Professor Godefroy.)—Si $x+y+z=0$, démontrer qu'on a

$$\left(\frac{y-z}{x}+\frac{z-x}{y}+\frac{x-y}{z}\right)\left(\frac{x}{y-z}+\frac{y}{z-x}+\frac{z}{x-y}\right)=9. \quad \ldots\ldots 122$$

10481. (Professor Cesáro.) — Soit $p_r=(1-x)(1-x^2)(1-x^3)\ldots(1-x^r)$. Démontrer l'identité $\frac{1}{p_n}-\frac{x}{p_1 p_{n-1}}+\frac{x^3}{p_2 p_{n-2}}-\ldots\pm\frac{x^{\frac{1}{2}n(n+1)}}{p_n}=1.$ 126

10486. (Professor de Longchamps.) — On considère deux cercles Δ, Δ', dont les centres sont les points O, O'. La droite OO' rencontre Δ, Δ', respectivement, aux points A, B; A', B'. Soient M, M' les extrémités de deux rayons mobiles, parallèles, et de même direction. Cela posé, AM, B'M' se rencontrent en P; BM, A'M' en Q. Démontrer que PQ passe constamment par un point fixe. 99

10488. (Professor Hendricks.)—The *d priori* probability that *ace* will be thrown at a single cast with a die is $\frac{1}{6}$, $= k$ say. A, whose veracity is p_1, says that B, whose veracity is p_2, said that *ace* was thrown at a single cast; find the probability that *ace* was thrown. [This Question was proposed in our columns many years ago, but Professor HENDRICKS thinks that none of the solutions then published are satisfactory.] ... 93

10489. (Professor Dépres.) — On donne dans un plan deux triangles ABC, A'B'C'; trouver le lieu d'un point P' tel que les perpendiculaires abaissées de A sur A'P', de B sur B'P', de C sur C'P' concourent en un même point P. Trouver également le lieu du point P..... 105

10499. (W. J. C. Sharp, M.A.)—A circle is just immersed in a fluid the density of which is known to vary as some function of the depth. It is found that the centre of pressure divides the vertical diameter into two parts, which are as 7 : 3; find the law of density. 96

10504. (For Enunciation, see Question 10467) 82

10508. (J. O'Byrne Croke, M.A.)—Taking the general equation of the parabola as $(\alpha x+\beta y)^2+2gx+2fy+c=0$,

prove that (1) the focus and (2) the directrix are

$$x=\left\{\beta(g^2-f^2)+\beta c(\alpha^2+\beta^2)-2\alpha fg\right\}/\left\{2(\alpha f-\beta g)(\alpha^2+\beta^2)\right\} \quad \ldots (1);$$
$$y=\left\{\alpha(g^2-f^2)-\alpha c(\alpha^2+\beta^2)+2\beta fg\right\}/\left\{2(\alpha f-\beta g)(\alpha^2+\beta^2)\right\}$$
$$\beta x-\alpha y+\left\{(\alpha^2+\beta^2)c-f^2-g^2\right\}/\left\{2(\beta g-\alpha f)\right\}=0 \quad \ldots\ldots (2).$$
..... 123

10509. (R. H. W. Whapham, B.A.) — In an epicycloid, let C be the centre of the fixed circle, C' that of the rolling circle; P any point on the epicycloid; O the corresponding point of contact of the two circles. Prove the following construction for the centre of curvature at P:—Produce CC' to K, such that C'K = CC'. Join KP. Through C draw CT parallel to KP, meeting PO in T; then T is the centre of curvature at P. 117

10510. (D. T. Griffiths.) — The three tangents to a parabola at the points A, B, C form an equilateral triangle. Prove that the ellipse which touches sides of the triangle ABC at their mid-points passes through the focus of the parabola. 115

10515. (Professor Catalan.) — Si les diagonales AD, BE, CF d'un hexagone ABCDEF sont égales, et que les côtés soient parallèles, deux à deux, l'hexagone est inscriptible à une circonférence. 108

10518. (Professor Schnevikoff.) — Soient A le sommet d'un tétraèdre, BCD sa base, G le point d'intersection des médianes du triangle BCD. Démontrer que $3AB^2 + 3AC^2 + 3AD^2 = BC^2 + BD^2 + CD^2 + 9AG^2$. ... 114

10519. (Professor Décamps.)—Dans tout quadrilatère circonscriptible, les points de rencontre des hauteurs des quatre triangles formés par deux côtés consecutifs et leur corde de contact avec le cercle sont les sommets d'un parallélogramme. 110

10526. (Professor Mathews.) — Two plane convex quadrilaterals ABCD, A'B'C'D' are equal in area, and each side of one is equal to the corresponding side of the other (AB = A'B', &c.) : prove that

$$AC \cdot BD = A'C' \cdot B'D'.$$ 128

10530. (Professor Emmerich.) — If ω be the Brocard-angle, and δ any acute angle of a triangle, prove that $\cot\omega \sin 2\delta > 2$ in an obtuse-angled triangle, and < 2 in an acute-angled triangle..... 112

10532. (Professor Orchard, M.A.) — Find the condition that we may solve, as a quadratic, $x^4 + 27x^3 + 9x^2 + 3x + e = 0$. 127

10534. (Rev. H. Sewell, B.A.) — If $a, b, c, \ldots k$ be n positive quantities, prove that $\left(\frac{a+b+c+\ldots+k}{n}\right)^{a+b+c+\ldots+k} < a^a b^b c^c \ldots k^k$. 107

10538. (W. J. C. Sharp, M.A.) — A flexible hawser is fastened to a point at the bottom of a stream, in which the force of the current varies as any function of the depth. Find the differential equation to the catenary in which it hangs, and show that, if the axis of y be vertical, m be the weight of the unit of length of the hawser when weighed in water, b an arbitrary constant, and T the tension, $T = (ms - b)\, ds/dy$. 111

10539. (Hilda Hudson.) — Through two points (A, B) draw a circle so that the end (H) of the diameter (GH) parallel to AB may be at a given distance from the point B. 109

10546. (F. R. J. Hervey.) — A conic has its axes parallel to the asymptotes of a rectangular hyperbola, and cuts it in four points. Prove that the mean of the intersections lies midway between the centres of the curves..... 126

APPENDIX.

Unsolved Questions 129

CORRIGENDA.

VOL. LII.

Page 47.—In the Solution of Quest. 3216, Mr. BIDDLE inadvertently omitted to divide the integral by the difference between the limits of θ; thus, the correct result, $\frac{2}{\pi}$ of that given, is $\frac{2}{\pi} - \frac{\pi}{8}$, or ·2439207, which is not quite $\frac{61}{250}$.

VOL. LIII.

Page 34.—*For* Question 8892 *read* 8898.

MATHEMATICS

FROM

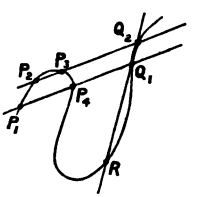
THE EDUCATIONAL TIMES.

WITH ADDITIONAL PAPERS AND SOLUTIONS.

2853. (Professor SYLVESTER.)—1. Let P be a given point in a cubic. Show that, in general, three conics may be drawn having contact of the 3rd order with the cubic at P and ordinary contact with it at some other point; but that no such conics (or only two) can be drawn when P is a pluperfect point of the 1st (and 2nd or 4th) order.

2. Also that, in general, twelve conics may be drawn having ordinary contact with the cubic at T, and contact of the 3rd order with it at some other point; but that only ten such conics can be drawn when T is a pluperfect point of the 2nd order.

Solution by Professor SCHOUTE.



1. Let P_1, P_2, P_3, P_4 be four points of the cubic, and let the lines P_1P_4, P_2P_3, Q_1Q_2 meet the curve in the third points Q_1, Q_2, R. Then R is the point opposite to $P_1P_2P_3P_4$; in other words, the conics through these four points meet the cubic in couples of points, the joins of which pass through R. And when the four points P_1, P_2, P_3, P_4 coincide in P and the two points Q_1, Q_2 in Q, Q and R are the first and second tangential point of P.

This proves the first part of the theorem. For through any point S of a cubic pass four tangents that touch it elsewhere. So the points T of contact of the three tangents through R, that differ from RQ, are the points of contact of the three conics that fulfil the conditions of the problem, the point Q itself corresponding to the improper solution consisting of the line PQ counted twice.

When P is a pluperfect point of the n^{th} order it is a point of contact of the order $3n-1$ of the cubic with a curve of the order n. So for $n=1$ P is a point of inflexion. For $n=2$ the tangential point Q of P is a point of inflexion. And for $n=4$ the second tangential point R of P is a point of inflexion.

If P is a point of inflexion, any conic having with the cubic a contact of the 3rd order degenerates in two lines; so there is no proper solution.

If Q is a point of inflexion, R coincides with Q. Then the two tangents through Q that differ from QP (corresponding to the conic with contact

of the 5th order at P), and from the inflexional tangent (corresponding to the line PQ counted twice), indicate the two solutions.

If R is a point of inflexion, the two tangents through R, which differ from RQ and the inflexional tangents, correspond to the two solutions.

2. If a conic and a cubic have a contact of the 3rd order in P and an ordinary contact at T, the second tangential point R of P coincides with the first tangential point of T. Through R pass three tangents RQ, that differ from RT, and each of the three points Q is the tangential point of four points P. So we find in all 12 points P, *i.e.*, 12 solutions.

When R is a point of inflexion, the two tangents RQ_1 and RQ_2, that differ from RT and the inflexional tangent, procure eight points P, and the inflexional tangent itself procures still two solutions for which P coincides with one of the points Q_1, Q_2. So we find, in all, ten undegenerating conics that answer to the given conditions.

Remarks.—I. The principal part of this theorem may be stated in the following popular form :—

"In a family, a father has four sons; show that every male member of this family has three uncles and twelve nephews."

The particularities of the pluperfect points of different order cannot be translated in this way, since it is impossible to be one's own son.

II. The twelve points P corresponding to a given point T are situated by four on 24 conics, that touch the cubic at T. When p, q, r, t represent the arguments of the points P, Q, R, T, and w_1, w_2 are the two periods, we see that the condition $4p + 2t \equiv 0 \pmod{w_1, w_2}$ is fulfilled by the suppositions

$$2p + q \equiv 0,\ 2q + r \equiv 0,\ 2t + r \equiv 0 \pmod{w_1, w_2},$$

in harmony with the solution we have given.

So the twelve points P are determined by the equation

$$p = \tfrac{1}{4}t + \tfrac{1}{4}(k_1 w_1 + k_2 w_2) \ldots (k_1 = 1, 2, 3, 4\ ;\ k_2 = 1, 2, 3, 4).$$

When the four combinations ($k_1 = 2, 4$; $k_2 = 2, 4$) that correspond to degenerating conics are excluded we indicate these twelve points by the symbols

	(1, 1),	(2, 1),	(3, 1),	(4, 1),
	(1, 2),		(3, 2),	
	(1, 3),	(2, 3),	(3, 3),	(4, 3),
	(1, 4),		(3, 4),	

and represent these by the numbers

1,	5,	9,	13,
2,		10,	
3,	7,	11,	15,
4,		12.	

Then each of the 24 quadruples,

1, 2, 9, 12,	1, 5, 7, 11,	2, 4, 10, 12,	3, 9, 13, 15,
1, 2, 10, 11,	1, 11, 13, 15,	2, 5, 7, 10,	4, 5, 7, 12,
1, 3, 5, 15,	2, 3, 9, 10,	2, 10, 13, 15,	5, 7, 13, 15,
1, 3, 7, 13,	2, 3, 11, 12,	3, 4, 9, 12,	5, 10, 12, 13,
1, 3, 9, 11,	2, 4, 5, 13,	3, 4, 10, 11,	7, 9, 11, 13,
1, 4, 9, 10,	2, 4, 7, 15,	3, 5, 7, 9,	7, 10, 12, 15,

consists of four points situated on a conic that touches the cubic in T. For in every combination the factors of k_1 and k_2 are divisible by four.

III. This problem is included in the following one:—

"Through two points A and B of a cubic pass sixteen conics, that have elsewhere a contact of the 3rd order with the cubic."

"The sixteen points of contact of these conics are situated by four on 116 conics, that pass through A and B."

For this extension compare *Wiskundige Opgaven*, tome iii., page 417, problem 187.

3707. (J. B. Saunders.)—A spherical air-bubble, having risen from a depth of 1000 feet in water, was one inch in diameter when it reached the surface; find its diameter at the bottom.

Solution by D. Biddle.

Ignoring any change in temperature, and assuming that, to the extent demanded, Boyle's law holds good, namely, that the volume of a given mass of air, kept at a given temperature, is inversely as the pressure, the required diameter $= (p/q)^{\frac{1}{3}}$, where $p \equiv$ the pressure per square inch on the bubble as it emerges at the surface, and $q \equiv$ the pressure per inch on the bubble at the bottom of the 1000 feet of water. If the volume were inversely as the *total* pressure on the spherical superficies, the diameter would be $(p/q)^{\frac{1}{2}}$, which is greater than $(p/q)^{\frac{1}{3}}$; but such is not the case. The pressure in a fluid being the same in all directions at a given point, it is easy to see that equilibrium can only be produced when the pressure inside and outside the bubble at any part of its surface is the same; and this, according to Boyle's law, can only occur when the diameter of the bubble is $(p/q)^{\frac{1}{3}}$, unless there be change of temperature.

But temperature is not the only disturbing element in the calculation; there are numerous causes of variation in the values of p and q. Both are influenced by the atmospheric pressure, which is ever changing; also by the distance of the place of observation from the earth's centre, which affects the weight not only of the air, but of the water. There is also the temperature of these, as well as of the bubble, to be taken into account. Moreover, the compressibility of the water, as affecting the amount in a column of 1000 feet, is an important factor; and if it be sea-water, instead of pure water, through which the bubble rises, the calculation is especially encumbered, for sea-water is at once heavier and less compressible.

Assuming, however, the truth not only of Boyle's law, but also of Charles' law, that the product of the volume and pressure of any gas is proportional to the absolute temperature, calculated from $-460°$ Fahrenheit; assuming that we know the temperatures of the air in the bubble at both places, and that the temperature of the water is constant throughout the course of the bubble; assuming, further, that the weight of a body above the surface of the earth is inversely as the square of its distance from the centre, but below the surface directly as its distance; that the compressibility of pure water is (within the given limits) about

·000048 per atmosphere of 14·7 lbs. to the square inch; and that the compressibility of sea-water is ·925 of that amount, or about ·000044 per atmosphere, whilst its specific gravity is 1·0263 (except in the case of the Dead Sea, where it is 1·2403); we may proceed as follows:—

Let $r \equiv$ the distance of the surface of the water from the centre of the earth; $t_1 \equiv$ the temperature of the bubble at the bottom, and $t_2 \equiv$ the temperature of it at the top of the water; $\alpha \equiv$ the height in feet of a column of water hermetically sealed, of section-area one inch, and weighing (when horizontal or collected at r) 14·7 lbs.; $\beta \equiv$ the height of a column of water rising from the neighbourhood of the emerged bubble, and supported exactly by the atmospheric pressure at the particular time and place. For convenience of calculation, let the water used in each case be similar to that concerned in the question. Then, approximately, the standard atmospheric pressure A is to the atmospheric pressure p, on the emerged bubble, as

$$\int_0^\alpha \frac{r^2}{(r+\alpha-x)^2} \cdot \frac{a^2 dx}{a^2 - \cdot 000048 \int_0^x \frac{r^2}{(r+\alpha-y)^2} dy} : \int_0^\beta \frac{r^2}{(r+\beta-x)^2} \cdot \frac{a^2 dx}{a^2 - \cdot 000048 \int_0^x \frac{r^2}{(r+\beta-y)^2} dy},$$

if pure water be used; and the same formula will serve for sea-water if ·000044 be substituted for ·000048, the real difference being, however, in α and β, not in the proportion of A to p. But, within the present limits, sufficient accuracy is obtained by the ordinary ratio, $\alpha : \beta = 14{\cdot}7 : p$, whence $p = 14{\cdot}7\,\beta/\alpha$.

We next have $p : q = \beta : \lambda$, where

$$\lambda = \beta + \int_0^{1000} \frac{r-x}{r} \cdot \frac{a^2 dx}{a^2 - \cdot 000048 \left(\beta + \int_0^x \frac{r-y}{r} dy \right)}$$

$$= \beta + \int_0^{1000} \frac{2a^2 (r-x)\, dx}{2r(a^2 - \cdot 000048\beta) - \cdot 000096 rx + \cdot 000048 x^2},$$

which reduces to

$$\beta + \frac{a^2}{\cdot 000048} \left[\log 2r(a^2 - \cdot 000048\beta) - \log \left\{ 2r(a^2 - 000048\beta) - \cdot 096r + 48 \right\} \right],$$

for pure water. For sea-water we obtain

$$\beta' + \frac{a'^2}{\cdot 000044} \left[\log 2r(a'^2 - \cdot 000044\beta') - \log \left\{ 2r(a'^2 - \cdot 000044\beta') - 088r + 44 \right\} \right].$$

Examples.

Col. Clarke ("Geodesy," p. 319) gives the semi-axes of the earth

$$a = 20926202, \quad c = 20854895,$$

and we can find r for any latitude θ from the formula

$$r = ac / (a^2 \sin^2\theta + c^2 \cos^2\theta)^{\frac{1}{2}}.$$

At Greenwich $\theta = 51° 21' 38\frac{1}{2}''$ and $r = 20882610$. Let the surface of the water in the question be at sea-level anywhere in this latitude, and the absolute temperature of the emerged bubble, $t_2 = (460 + 60)°$ F., whilst $t_1 = (460 + 75)°$ F.; also let $\alpha = 33{\cdot}35$, $\beta = 32{\cdot}90$, in the case of fresh water. Then, in the case of sea-water, $\alpha' = 32{\cdot}50$, $\beta' = 32{\cdot}057$. We also at once obtain $p = 14{\cdot}502$ lbs. In the case of fresh water,

$\lambda = 32{\cdot}90 + 23171302\,(\log 46452159574 - \log 46450154891) = 1032{\cdot}9335.$

But with the same conditions in the case of sea-water,

$\lambda' = 32{\cdot}06 + 24005682\,(\log 44107504286 - \log 44105666776) = 1031{\cdot}7750.$

In the former case, $q = 455{\cdot}307$ lbs., and $p/q = {\cdot}0318510$. In the latter case, $q' = 465{\cdot}625$ lbs., and $p/q' = {\cdot}0310698$. But for the difference in compressibility, q' should have been $466{\cdot}9$ lbs.

Now $t_1/t_2 = 535/520 = 1{\cdot}028846$. Consequently, in the case of fresh water, the required diameter

$$= \{({\cdot}0318510)(1{\cdot}028846)\} = {\cdot}3200059,$$

and in the case of sea-water,

$$\{({\cdot}0310698)(1{\cdot}028846)\}^{\frac{1}{3}} = {\cdot}3173677.$$

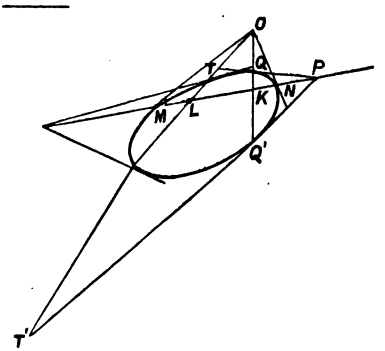
If the temperature of the bubble remained the same, the required diameter would in the respective cases be, ·3169868 and ·3143735 of an inch.

10237. (D. T. Griffiths.)—The pairs of tangents to a conic from points on a straight line determine an involution on any transversal through its pole, or on any tangent to the conic.

Solution by Emily Perrin; G. G. Storr, M.A.; *and others*.

Let MN be a fixed line, OL any line through its pole O, P any point on MN, and let PQ, PQ′, the tangents from P, meet OL in T, T′.

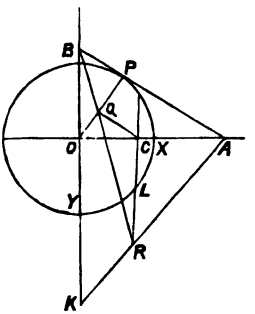
The pencil P{OQNQ′} is harmonic, therefore {OTLT′} is harmonic; and the points O, L are fixed, therefore the pairs of points L, L′ determine an involution on OL.



10226. (Professor Mandart.)—On donne un cercle O, deux diamètres rectangulaires OX, OY, et une perpendiculaire CL en un point C de OX. La tangente menée au point P de la circonférence rencontre OX en A,

OY en B ; la perpendiculaire élevée en A sur AB coupe CL en R. Démontrer que le lieu géométrique du point d'intersection des droites OP, BR est la circonférence décrite sur OC comme diamètre.

Solution by G. G. Morrice, M.A., M.B. ; H. J. Woodall ; *and others.*



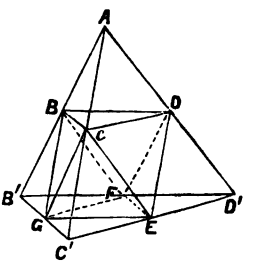
Let AR, OY intersect in K, OP, RB in Q. Then CR is parallel to OY, and OQ to RK, therefore

$$OQ : OP = KR : KA = OC : OA,$$

therefore QC is parallel to AP, and therefore $\angle OQC$ is a right angle.

7856. (W. J. C. Sharp, M.A.)—Construct an octahedron such that it shall have two parallel faces equal and similar to each of the faces of a given tetrahedron, and show that its volume is four times that of the tetrahedron.

Solution by Professor Schoute.



Let ABCD be the given tetrahedron. Prolong AB, AC, AD and make

$$BB' = AB, \quad CC' = AC,$$
$$DD' = AD.$$

Then the joins of the mid-points B, C, D, E, F, G of the edges of the tetrahedron AB'C'D' are the edges and diagonals of the octahedron in question, &c.

[The volume of the whole tetrahedron AB'C'D' is eight times that of ABCD, and each of the four tetrahedra which must be cut from it to leave the octahedron is equal to ABCD, therefore the volume of the octahedron is four times that of ABCD.]

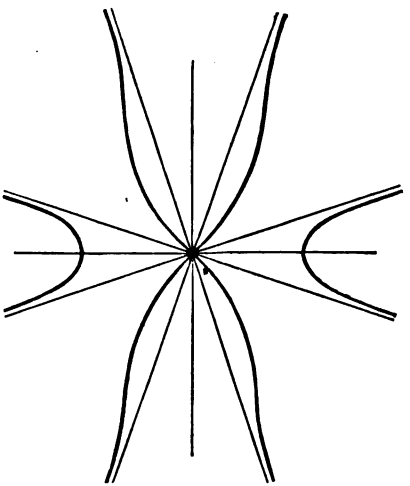
7850. (Asparagus.) — An equilateral triangle is inscribed in the rectangular hyperbola $x^2 - y^2 = a^2$; prove that the locus of the middle points of its sides is the quartic $3(x^2+y^2)^2 = 4(x^2-y^2)(x^2-y^2-a^2)$, or in polar coordinates $r^2 = \frac{a^2 \cos 2\theta}{1 + 2\sin 2\theta}$; and trace the curve.

Solution by Professor WOLSTENHOLME, M.A., Sc.D.

The curve is as given in the accompanying figure. The asymptotes all pass through the centre of the hyperbola, and are inclined at angles

$$\pm\frac{\pi}{12}, \quad \pm\frac{5\pi}{12}$$

to the transverse axis. The two former contain two hyperbolic branches, vertices at distance $2a$ from the centre; while the latter touch at ∞ two sinuous branches which have inflexions at the centre, the tangents at the inflexions being the asymptotes of the hyperbola.

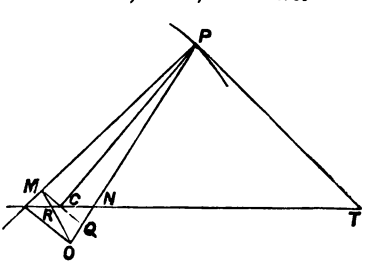


10017. (R. KNOWLES, B.A.)—The circle of curvature is drawn at a point P of a conic, M is the mid-point of the common chord; the diameter of the conic through M meets the normal at P in Q; the normal meets the transverse axis in N; O is the centre of curvature at P. Prove that QN : NP = OQ : OM.

Solution by W. S. FOSTER; G. G. STORR, M.A.; *and others.*

Let OM meet the axis in R. Since PM, PT are equally inclined to the axis; and the tangent at the extremity of CM is parallel to PM, therefore CM, CP are equally inclined to the axis, therefore CN bisects the angle PCQ,

$\therefore$ QN : NP = CQ : CP.



Also OM, OP are equally inclined to the axis, *i.e.*,

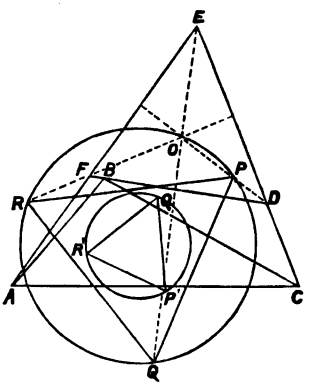
ONR = ORN,

therefore OMQ = ORN − MCR = ONR − NCP = CPQ,

therefore OQ : OM = CQ : CP = QN : NP.

10263. (Professor MORLEY.)—Equilateral triangles have sides passing through given points; prove that the centre locus is two concentric circles, coincident when points collinear.

Solution by Professor SCHOUTE.



When the sides FD and DE of the equilateral triangle DEF pass respectively through B and C, the bissectrix DO of the angle D passes through the points P that bisect the minor arc BC of the circle BCD. As there are two circles through B and C, that contain an angle of 60°, there are two points P that may be distinguished as P and P'; these points lie symmetrically with respect to BC, and are the vertices of two isosceles triangles with the base BC and a vertical angle of 120°.

When the points Q and Q' have the same significance with respect to CA as P and P' have with respect to BC, it is immediately evident that the sides of the angle DOE measuring 120° pass respectively through one point of the pairs P, P' and Q, Q'. So the locus of O consists of circles. When R and R' are for AB what P and P' are for BC, when as in the diagram P, Q, R are the vertices of the outward and P', Q', R' those of the inward isosceles triangles, and due attention is paid to the sign of the angles, it appears that the locus consists of the two circles PQR and P'Q'R', and that the triangles PQR and P'Q'R' are equilateral. 10263

Finally, the concentricity of the two circles can be demonstrated by means of the following theorem of Laisant. When on the sides of a polygon $A_1A_2 \dots A_n$ directly similar triangles $A_1B_1A_2$, $A_2B_2A_3 \dots A_nB_nA_1$ are described, the two systems of equally loaded points $A_1, A_2 \dots A_n$, and $B_1, B_2 \dots B_n$ have the same centre of gravity. For this theorem proves that the triangles PQR and P'Q'R' correspond in centre of gravity with ABC, and therefore with one another too.

When A, B, C are collinear the concentric circles are symmetrical with respect to ABC; in this case they coincide.

One may compare Laisant's communication (*Annuaire de l'Association française*, Congrès de Havre, 1877), and *Wiskundige Opgaven* (t. 3, pp. 48 and 52).

10275. (D. T. GRIFFITHS.)—The perpendiculars from the vertices on the sides of the triangle ABC meet the sides of the pedal triangle DEF at the points *d*, *e*, *f*. If AB and DE, AC and FD, BC and FE intersect at the points X, Y, Z respectively, prove that each side of triangle *def* passes through one of three points X, Y, Z.

Solution by A. J. Pressland, M.A.; R. Knowles, B.A.; *and others.*

Since the angles FED, DEZ are bisected by E*e*, EC;

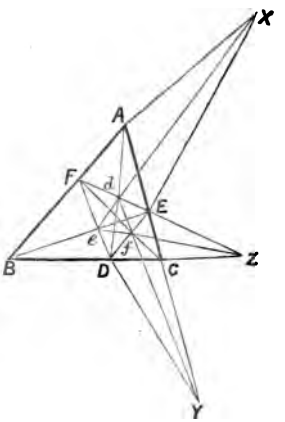
therefore $$\frac{Fe}{eD} = \frac{FE}{ED},$$

and $$\frac{DC}{CZ} = \frac{DE}{EZ};$$

therefore $$\frac{Fe}{eD} = \frac{FE \cdot CZ}{EZ \cdot CD};$$

hence, by the converse of Ceva's theorem, FC, DE, Z*e* are concurrent, therefore *ef* passes through Z.

In the same way it may be shown that *ed* and *df* pass through points X and Y respectively; hence, each side of triangle *def* passes through one of the three points X, Y, Z.



7668. (Professor Sylvester)—If m be any positive integer, and a, b, c any real quantities whatever, prove that the equation

$$a\left(1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\ldots+\frac{x^m}{1.2.3\ldots m}\right)$$
$$+b\left(1-x+\frac{x^2}{1.2}-\frac{x^3}{1.2.3}+\ldots\pm\frac{x^m}{1.2.3\ldots m}\right)+c=0$$

can never have more than two real roots.

Solution by Professor Schoute.

The equation can be thrown into the form

$$(a+b)\left(1+\frac{x^2}{1.2}+\frac{x^4}{.2\ \ 3.4}+\ldots\right)+(a-b)\left(\frac{x}{1}+\frac{x^3}{1.2.3}+\ldots\right)+c=0,$$

which may be represented by

$$pU_m+qV_{m-1}+c=0,\ m \text{ even}, \qquad pU_{m-1}+qV_m+c=0,\ m \text{ odd}.$$

Now we suppose the problem to be true for m, and prove it for $m+1$. Then, as it is true for $m=2$, it will be true in general. But at that end we consider the two cases, m even and m odd, apart from one another.

For m even we must prove that the curve $y=pU_m+qV_{m+1}+c$ meets the axes of x in only one point. Now $dy/dx=pV_{m-1}+qU_m$ vanishes for two real values of x, or for none. In the latter case y has neither maximum nor minimum, and is infinite of different sign for $x=-\infty$ and $x=+\infty$. So the continuous function y will disappear for one and only one real value of x. In the first case the two real roots of $dy/dx=0$ have the same sign; for $pV_{m-1}+qU_m$ has the sign of q in the three suppositions

$x = -\infty, 0, \infty$. So the maximum point and the minimum point of y lay on the same side of the axis of x, and y is infinite with opposite signs for $x = -\infty$ and $x = +\infty$; which also proves y to be zero for one and only one real value of x.

For m odd we have $y = pU_{m+1} + qV_m + c$ and $dy/dx = pV_m + qU_{m-1}$. Now dy/dx vanishes for one real value of x. So y has one maximum or minimum. And in that case y is zero for two (or none) real values of x.

In the cases $p = 0$, or $q = 0$, that seem to form exceptions as to the demonstration, the theorem is self-evident.

10356. (Benjamin Williamson, F.R.S.) — Given an elementary proof of Stokes' theorem, viz.—If a portion of a surface S be bounded by a closed curve s, and if u, v, w be continuous functions of x, y, z, then

$$\left\{\iint l\left(\frac{dw}{dy} - \frac{dv}{dz}\right) + m\left(\frac{du}{dz} - \frac{dw}{dx}\right) + n\left(\frac{dv}{dx} - \frac{du}{dy}\right)\right\} dS = \int (u\,dx + v\,dy + w\,dz),$$

where l, m, n are the direction cosines of the normal to the surface at the point xyz, and the double integral is taken over the surface, and the single integral along its boundary in the positive direction.

Solution by W. J. Johnston, M.A.

z and w are evidently functions of x and y; hence, round curve

$$\int w\,dz = \int dw\,dz \text{ over portion of surface included} = \iint \frac{d(z, w)}{d(x, y)}\,dx\,dy$$

$$= \iint \left(p\frac{dw}{dy} - q\frac{dw}{dx}\right) dx\,dy$$

$$= \iint \left(\frac{p}{(1+p^2+q^2)^{\frac{1}{2}}}\frac{dw}{dy} - \frac{q}{(1+p^2+q^2)^{\frac{1}{2}}}\frac{dw}{dx}\right)(1+p^2+q^2)^{\frac{1}{2}}\,dx\,dy$$

$$= \iint \left(l\frac{dw}{dy} - m\frac{dw}{dx}\right) dS.$$

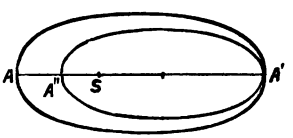
The theorem is proved by adding to this similar equations for u, v.

8892. (Professor Asûtosh Mukhopâdhyây, B.A., F.R.A.S.)—Two imperfectly elastic particles, whose coefficient of elasticity is λ, are projected with equal velocities in opposite directions, at right angles to the line of force at the point of projection; assuming the force to vary inversely as the square of the distance, and the second impact to take place midway between the point of projection and the centre of force, show that (1) $e_1 = \pm(2-3\lambda^2)/\lambda^2$; (2) $e_2 = \pm(3-4\lambda^2)$,

where e_1 is the eccentricity of the first orbit, and e_2 that of the orbit between the first and second impacts. Hence, determine when the eccentricity is unaltered by the impact.

Solution by Rev. T. Galliers, M.A.; G. G. Storr, M.A.; *and others.*

Let S be the centre of force; A, A′, A″ the points of projection first and second impact respectively; and $2a_1$, $2a_2$ major axes of first and second orbits.



Consider the case in which

$$SA = a_1(1-e_1),$$

and $\therefore$ $SA' = a(1+e_1)$,

and $SA' = a_2(1+e_2)$, and therefore $SA'' = a(1-e_2)$

Now $2SA'' = SA'$, therefore $2a_2(1-e_2) = a_1(1-e_1)$ .....(1),

also $a_1(1+e_1) = SA' = a_2(1+e_2)$ .....(2).

If v = velocity at A′, $v^2 = \mu(1-e_1)/\{a_1(1+e_1)\}$.....(3),

also $\lambda^2 v^2 = \mu(1-e_2)/\{a_2(1+e_2)\}$.....(4).

From (1), (2), (3), (4),

$$1-e_2 = \lambda^2(1-e_1), \qquad 1-e_1e_2 = 3(e_2-e_1).$$

Hence $e_1 = (2-3\lambda^2)/\lambda^2$. $e_2 = 3-4\lambda^2$.

Considering the other possible cases, we shall get as the final result

$$e_1 = \pm(2-3\lambda^2)/\lambda^2, \quad e_2 = \pm(3-4\lambda^2) \text{.....(5).}$$

If, in (5), the two upper signs be taken together, $e_1 = e_2 = 1$, when $\lambda = 1/\sqrt{2}$. If the signs in (5) be taken appropriately, $e_1 = e_2 = 3-2\sqrt{2}$, when $\lambda = 1/\sqrt[4]{2}$.

10345. (Professor de Longchamps.)—Une ellipse étant rapportée à ses axes, démontrer que les coordonnées d'un point quelconque de la normale à l'ellipse, en un point dont les coordonnées sont $a\cos\phi$, $b\sin\phi$, peuvent être représentées (λ étant paramètre variable) par les formules

$$x = \frac{c^2\cos\phi}{a} \cdot \frac{\sin\lambda\sin\frac{1}{2}(\lambda+\phi)}{\cos\frac{1}{2}(\lambda-\phi)}, \quad y = -\frac{c^2\sin\phi}{a} \cdot \frac{\cos\lambda\cos\frac{1}{2}(\lambda+\phi)}{\cos\frac{1}{2}(\lambda-\phi)}.$$

Solution by J. J. Barniville; R. Whapham, B.A.; *and others.*

We have

$$\frac{ax}{\cos\phi} = \frac{c^2\sin\lambda\sin\frac{1}{2}(\lambda+\phi)}{\cos\frac{1}{2}(\lambda-\phi)}, \quad \frac{by}{\sin\phi} = c^2\frac{\cos\lambda\cos\frac{1}{2}(\lambda+\phi)}{\cos\frac{1}{2}(\lambda-\phi)};$$

therefore

$$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = c^2;$$

i.e., the equation of the normal, c^2 being $= a^2-b^2$.

9480. (A. R. Johnson, M.A.)—Show that, if $O = 0$ be the condition that a pair of the lines $f(x, y) \equiv (a_0, a_1, a_2 \ldots a_n)(x, y)^n$, should be

at right angles, and R denote the resultant of $f(x, y)$ and $f(-y, x)$, then

$$O = [R/\{f[1, (-1)^{\frac{1}{4}}] f[1, -(-1)^{\frac{1}{4}}]\}]^{\frac{1}{2}},$$

so that O is of order $n-1$ in the coefficients. [The axes are supposed to be rectangular.]

Solution by Professor SEBASTIAN SIRCOM, M.A.

Forming the resultant in the usual way, we shall have, since the roots of $f(-y, x)$ are the reciprocals with their signs changed of those of $f(x,y)$

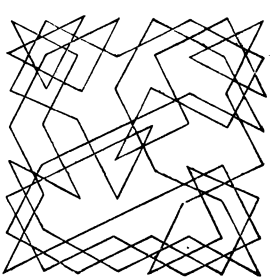
$$\begin{aligned} R &= a_0^n a_n^n \left(\alpha+\frac{1}{\alpha}\right)\left(\alpha+\frac{1}{\beta}\right) \dots \left(\beta+\frac{1}{\alpha}\right)\left(\beta+\frac{1}{\beta}\right) \dots \\ &= a_0^{2n}(1+\alpha^2)(1+\beta^2) \dots (\alpha\beta+1)^2 \dots \\ &= a_0\{1-(-1^{\frac{1}{4}})\alpha\} \dots a_0\{1+(-1)^{\frac{1}{4}}\alpha\} \dots a_0^{2n-2} O^2 \\ &= f\{1, (-1)^{\frac{1}{4}}\} f\{1, -(-1)^{\frac{1}{4}}\} a^{2n-2} O^2, \end{aligned}$$

which gives the result.

10144. (Professor NASH.)—Assuming that a knight's tour is completely determined when the first and last squares are given; show that there are 136 distinct tours, no one of which is the reflected image of another, or can be derived from another by rotating the board. In 21 of these tours the terminal squares are a knight's move apart; if these squares be joined, show that a tour can be constructed from which each of the remaining 20 re-entrant tours can be derived either by rotation of the board or by reflection. Hence the 136 tours are reduced to 116.

Solution by the PROPOSER.

It is clear that any square and its image with respect to either diagonal are of the same colour, and that a square and its image with respect to either of the lines bisecting the board, and parallel to its sides, are of opposite colours. There are altogether 32^2 tours commencing on a white square. In the first diagram the 32 possible initial squares are divided into three groups, indicated by the letters P, Q, R. Those marked Q are the images of those marked P with respect to the diagonal BD, and the R's are the images of the P's and Q's with respect to the diagonal AC.



The final square being undetermined, this shows that all possible tours can be obtained by taking as initial square one of the 10 P's. This reduces the number to 10×32.

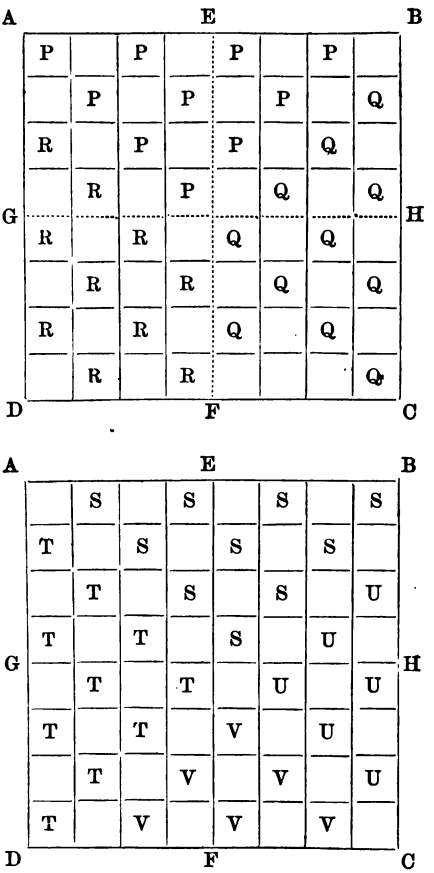
In the second diagram the final black squares are divided into four groups of 10, 10, 6 and 6 squares respectively, indicated by the letters S, T, U, V.

The S's are the images of the 10 initial squares with respect to the line EF. Hence of the 10^2 tours of the set PS, there are 10 in which the initial and final squares are images of one another, and of the remaining 90 each appears twice. so that there are only $55 (= 10 + \frac{90}{2})$ distinct tours of the form PS.

The T's are the images of the S's with respect to AC; the four P's lying along AC are their own images, and therefore, combined with any of the T's, give a tour which is the image of one of the tours PS.

The remaining six P's, with the ten T's, give 60 new tours.

By turning the board through a right angle in the direction CBA, any tour PM is transformed into a tour TP or PT, so that no new tour is obtained from the six U's.



The six V's are the images of the U's with respect to AC, and therefore give no new tours with the four P's which lie on AC. They are also the images of the other six P's with respect to GH, and therefore, as in the first case (PS), the number of different tours is reduced from 6^2 to $6 + \frac{30}{2} = 21$.

Hence the total number of distinct tours is $55 + 60 + 21 = 136$.

2. Adopting the system of notation in which the first figure indicates

the row, and the second the column, the following tour includes all possible re-entrant tours :—

11, 23, 42, 63, 51, 72, 84, 76, 88, 67, 55, 34, 15, 27, 48, 36, 44, 52, 71, 83, 75, 87, 68, 56, 77, 85, 66, 58, 37, 18, 26, 38, 17, 25, 46, 54, 35, 47, 28, 16, 24, 12, 31, 43, 64, 45, 53, 61, 82, 74, 86, 78, 57, 65, 73, 81, 62, 41, 22, 14, 33, 21, 13, 32, 11.

10354. (G. F. Howse, M.A.)—Evaluate the determinant

$$\begin{vmatrix} c & a & a & a & a & \dots \\ b & c & a & a & a & \dots \\ b & b & c & a & a & \dots \\ b & b & b & c & a & \dots \\ \dots & \dots & \dots & \dots & & \\ \dots & \dots & \dots & \dots & & \end{vmatrix}$$

of the nth order, all the terms in the diagonal being c, those to the right being a, those to the left b.

Solution by F. R. J. Hervey ; Prof. Curtis ; *and others.*

Subtract each row from the next ; the minor corresponding to any constituent of the first row then reduces to its first element. For, either we have on one side of the diagonal cyphers only ; or, if not, at the point where they fail on the left, the same condition is supplied on the right, in the manner shown. Hence the value of the determinant is

$$\begin{vmatrix} \dots & b-c & c-a & 0 & 0 & \dots \\ \dots & 0 & b-c & 0 & 0 & \dots \\ \dots & 0 & 0 & c-a & 0 & \dots \\ \dots & 0 & 0 & b-c & c-a & \dots \end{vmatrix}$$

$$c(c-a)^{n-1} - a\left\{(b-c)(c-a)^{n-2} - (b-c)^2(c-a)^{n-3} + \dots \pm (b-c)^{n-1}\right\}$$

$$= \left\{b(c-a)^n - a(c-b)^n\right\} / (b-a)$$

$$= c^n - ab\left\{C_{n,2}c^{n-2} - C_{n,3}c^{n-3}(b+a) + C_{n,4}c^{n-4}\frac{b^3-a^3}{b-a} - \dots \pm \frac{b^{n-1}-a^{n-1}}{b-a}\right\}.$$

10299. (Professor Neuberg.)—On projette une figure F_α, contenue dans le plan α, sur un second plan β ; soit F_β la projection. On projette, ensuite, F_β sur un plan γ faisant avec β le même angle que le plan α, les traces de α et γ sur β étant perpendiculaires. Démontrer que F_γ est semblable à F_α.

Solution by Rev. T. R. Terry, M.A. ; Prof. Emmerich, Ph.D. ; *and others.*

Let OX and OY be the traces of α and γ on β, and let θ be the angle of projection ; then F_α may be got from F_β by increasing lines parallel to OY in the ratio $\sec\theta : 1$. And F_γ may be got from F_β by decreasing lines parallel to OX in the ratio $1 : \cos\theta$. Therefore F_γ is similar to F_α, and corresponding lines in F_γ and F_α are in the ratio $1 : \cos\theta$.

10201. (H. L. ORCHARD, M.A., B.Sc.)—Find the relation between b and c, in order that the expression $\Sigma \equiv x^4 + x^3 + bx^2 + cx + 1 = 0$ may be exhibited as the product of two quadratic factors.

Solution by D. J. GRIFFITHS; H. J. WOODALL; *and others.*

Assume $\Sigma \equiv (x^2 + \frac{1}{2}x + \mathrm{P})(x^2 + \frac{1}{2}x + 1/\mathrm{P})$

$\equiv x^4 + x^3 + x^2(\mathrm{P} + 1/\mathrm{P} + \frac{1}{4}) + \frac{1}{2}x(\mathrm{P} + 1/\mathrm{P}) + 1;$

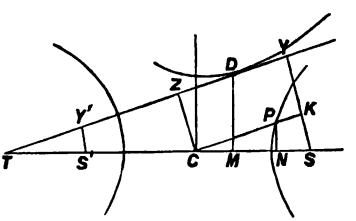
therefore $b = \mathrm{P} + 1/\mathrm{P} + \frac{1}{4}, \quad c = \frac{1}{2}(\mathrm{P} + 1/\mathrm{P});$

whence, eliminating P, the required relation is $b - 2c = \frac{1}{4}$.

10214. (Rev. T. W. OPENSHAW, M.A.)—Prove that the sum of the squares of the perpendiculars on the tangent to an hyperbola from the foci of the conjugate hyperbola is constant.

Solution by J. J. BARNIVILLE; R. KNOWLES, B.A.; *and others.*

Let SY, S′Y′ be perpendiculars from foci on tangent to conjugate at D; CZ perpendicular from centre; CP diameter parallel to tangent. Let tangent at D meet transverse axis in T, meeting SY in K; draw ordinates DM, PN.



$\mathrm{SY} + \mathrm{S'Y'} = 2\mathrm{CZ},$

$\mathrm{SY} - \mathrm{S'Y'} = 2\mathrm{SK},$

$\therefore \quad \mathrm{SY}^2 + \mathrm{S'Y'}^2 = 2(\mathrm{CZ}^2 + \mathrm{SK}^2),$

$\mathrm{SK}^2 + \mathrm{CZ}^2 : \mathrm{CS}^2 + \mathrm{CT}^2 = \mathrm{CZ}^2 : \mathrm{CT}^2 = \mathrm{PN}^2 : \mathrm{CP}^2.$

But $\mathrm{CT} : \mathrm{CA} = \mathrm{CA} : \mathrm{CM} = \mathrm{BC} : \mathrm{PN},$

therefore $\mathrm{CT}^2 + \mathrm{CA}^2 : \mathrm{CA}^2 = \mathrm{BC}^2 + \mathrm{PN}^2 : \mathrm{PN}^2,$

and $\mathrm{AC}^2 : \mathrm{BC}^2 = \mathrm{CN}^2 : \mathrm{BC}^2 + \mathrm{PN}^2,$

therefore $\mathrm{CT}^2 + \mathrm{CA}^2 : \mathrm{BC}^2 = \mathrm{CN}^2 : \mathrm{PN}^2,$

therefore $\mathrm{CT}^2 + \mathrm{CS}^2 : \mathrm{BC}^2 = \mathrm{CP}^2 : \mathrm{PN}^2 = \mathrm{CT}^2 + \mathrm{CS}^2 : \mathrm{SK}^2 + \mathrm{CZ}^2,$

therefore $\mathrm{SK}^2 + \mathrm{CZ}^2 = \mathrm{BC}^2, \quad \text{and} \quad \mathrm{SY}^2 + \mathrm{S'Y'}^2 = 2\mathrm{BC}^2.$

[*Otherwise*:—The equation to the tangent is $y - mx - (a^2m^2 - b^2)^{\frac{1}{2}} = 0$; the lengths of perpendiculars thereon from $(0, ae)$, $(0, -ae)$ are

$$\{ae - (a^2m^2 - b^2)^{\frac{1}{2}}\}/(1 + m^2)^{\frac{1}{2}}, \quad -\{ae + (a^2m^2 - b^2)^{\frac{1}{2}}\}/(1 + m^2)^{\frac{1}{2}},$$

and the sum of their squares $= 2a^2$. Hence the envelope of a straight line, the sum of the squares of the perpendiculars on which from two fixed points is constant, is an hyperbola. The theorem is a particular case of the general proposition, that if the sum of the squares of the perpendiculars from any number of fixed points on a straight line be constant, the line will envelope a conic, which can be readily deduced from the general

equation of envelopes of the second order, $Q^2 - PR = 0$. The conic would probably be an hyperbola in the case of all even numbers. If the difference of squares or product of perpendiculars be given, the envelope is also a conic, being in the former case a parabola.]

10129. (W. J. C. SHARP, M.A.)—Prove that any quadric, described through the six points in which the edges of a tetrahedron are each met by the plane through the opposite edge and a fixed point, will intersect the edges again in their six points of intersection with a system of planes through the opposite edges and another fixed point.

Solution by W. S. FOSTER; Professor AIYAR, M.A.; *and others.*

Taking the tetrahedron as that of reference, let

$$A\alpha^2 + B\beta^2 + C\gamma^2 + D\delta^2 + A'\alpha\beta + \ldots = 0 \text{ be any quadric,}$$

and let $\alpha = k_1\beta$, $\beta = k_2\gamma$, $\gamma = k_3\delta$, $\alpha = k_4\delta$, $\beta = k_5\delta$, $\gamma = k_6\alpha$,

$\alpha = k_1'\beta$, $\beta = k_2'\gamma$, $\gamma = k_3'\delta$, $\alpha = k_4'\delta$, $\beta = k_5'\delta$, $\gamma = k_6'\alpha$,

be planes through the edges of the tetrahedron. If these meet the opposite edges in the same points as the quadric,

$k_1k_1' = B/A$, $k_2k_2' = C/B$, $k_3k_3' = D/C$, $k_4k_4' = D/A$, $k_5k_5' = D/B$, $k_6k_6' = A/C$.

If the first six planes meet in a point, we have

$$k_1k_2k_3 = k_4,\ k_2k_3 = k_5,\ k_1k_2k_6 = 1;$$

therefore $$k_1'k_2'k_3' = k_4',\ k_2'k_3' = k_5',\ k_1'k_2'k_6' = 1;$$

therefore the other six planes meet in a point.

[This question is a particular case of a general property for space of n dimensions, proved in a paper "On Simplicissima," by Mr. SHARP, in the *Proc. Lond. Math. Soc.*, Vol. XVIII., p. 442. It is the analogue of a well-known theorem in Plane Geometry, given in TOWNSEND'S *Modern Geometry*, Vol. I., p. 180. See also Question 7585 on p. 124 of Vol. XLII. of our series.]

10318. (Prof. GUICHARD.)—Etant donné un triangle ABC, détermine dans son plan un point P tel qu'en menant la sécante PA'B' qui coupe AC en A' et BC en B', le point d'intersection M de AB', A'B décrive un cercle quand la sécante tourne autour du point P.

Solution by D. T. GRIFFITHS; Prof. SCHOUTE; *and others.*

When P is taken at random, the locus of M is the conic through A, B, C, that is touched in B and C by PB and PC. If P be the pole of BC with reference to the circumcircle of ABC, the locus of M will be the circumcircle.

2883. (Professor SYLVESTER.)—If there be two coincident solutions of $x+y+z=a+b+c$, $ax+by+cz=(a+b+c)^2$, $xyz=ayz+bzx+cxy$, show (1) that a curve can be formed of which a, b, c may be regarded as the trilinear coordinates, and determine its order; and find (2) how many (if any) points of the curve all three solutions will coincide; also (3) whether it is possible for the coordinates at any point of the curve to be commensurable quantities.

Solution by Professor SCHOUTE.

1. According to the problem, the surface $xyz=ayz+bzx+cxy$ of the third order, of which the origin is a node, is to be touched by the line $x+y+z=a+b+c$, $ax+by+cz=(a+b+c)^2$, or, in geometry of two dimensions, the line

$$(b+c)x+(c+a)y+(a+b)z=0 \quad \ldots\ldots (1)$$

is to touch the cubic $ayz(y+z)+bzx(z+x)+cxy(x+y)=0$ ..... (2).

Now the tangential equation of the cubic

$$p_x^3 \equiv (p_1x+p_2y+p_3z)^{(3)}=0 \quad \ldots\ldots (3)$$

is

$$(pqu)^2(rsu)^2(pru)(qsu)=0 \quad \ldots\ldots (4),$$

where q, r, s are repetitions of the symbols p. And as (4) is a Homogen of the fourth order in the coefficients of (2), and a Homogen of the sixth order in the tangential coordinates u, v, w, the equation obtained by substituting $b+c$, $c+a$, $a+b$ for u, v, w in (4) will be a Homogen of the order ten in a, b, c. So the locus in question is a curve of the tenth order.

2. Let the result of the elimination of x between (1) and (2) be represented by $Ay^3+3By^2z+3Cyz^2+Dz^3=0$, where A, B, C, D are Homogens of the third order in a, b, c. Now this equation has three equal roots under the two conditions $\dfrac{A}{B}=\dfrac{B}{C}=\dfrac{C}{D}$.

So the points of the curve of the tenth order, that correspond to three coincident solutions [for which (1) is an inflexional tangent of the cubic], are the points common to the two curves

$$AC-B^2=0, \quad BD-C^2=0$$

of the sixth order, diminished by the improper solutions $B=0$, $C=0$. Their number, therefore, cannot exceed 27.

3. The line (1) is a part of the cubic (2) in any of the three pairs of suppositions $a=0$, $b+c=0$; $b=0$, $c+a=0$; $c=0$, $a+b=0$. So the points $(0, 1, -1)$, &c. belong to the curve of the tenth order.

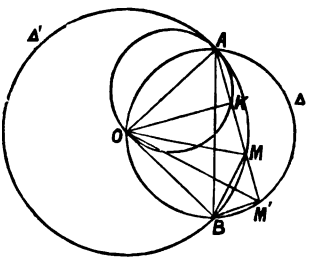
If the PROPOSER wishes to know three commensurable non-zero values of a, b, c, the solution of the last part is to be rejected.

10350. (Professor SOLLERTINSKI.)—Une circonférence Δ, passant par le centre O d'une autre circonférence Δ', coupe celle-ci en A, B. Une corde AM de Δ' rencontre Δ en M'. Démontrer (1) que BM'= MM'; et

(2) déduire une solution du problème suivant :—Deux points A, B, étant donnés sur une circonférence Δ, trouver sur Δ un point M′ tel que M′A + M′B soit égale à une longueur donnée $2h$.

Solution by R. H. W. WHAPHAM, B.A.; R. KNOWLES, B.A.; *and others.*

1. $\angle BOM' = BAM = \frac{1}{2}BOM$,
therefore $\angle BOM' = MOM'$,
and $OB = OM$,
therefore $BM' = MM'$.



2. Let Δ be the given circle; bisect the arc AB at O; with centre O and radius OA or OB describe a circle Δ′. On OA′ as diameter describe another circle AKO. Through A draw a line AKMM′ meeting Δ′ in M and Δ in M′; and such that $KM' = h$. Then M′ will be the required point on Δ. For AKO is a right angle, therefore K is the mid-point of AM; therefore

$$M'A + M'B = M'A + MM' \text{ by } (1) = 2KM + 2MM' = 2KM' = 2h.$$

3792. (G. S. CARR, M.A.)—A tube of smaller bore is joined to the open end of a common wheel-barometer tube, and alcohol is then poured in upon the mercury. Taking area of upper surface of mercury $= a$, that of the alcohol $= c$, area of common surface $= b$, sp. gr. of mercury $= m \times$ sp. gr. of alcohol; deduce the variation in the level of (σ) corresponding to a variation (x) in the ordinary barometric column of mercury.

Solution by D. BIDDLE.

Here we have $BC = m \cdot DE$, and AE = atmospheric pressure. Let $ay = bz = cu$, where y, z, u indicate relative change of level when disturbance occurs. Then

$$u - z + BC = m\,(DE)',$$

where $(DE)'$ represents the portion of mercury counterbalancing the alcohol after the given disturbance, as DE did before;

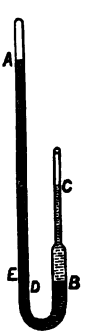
also $AD - y - z = AE - x + (DE)' = AE - x + (u - z + BC)/m$,

whence $m\,(x - y - z) = u - z$,

and $u = abmx/\{a\,(b-c) + (a+b)\,mc\}$.

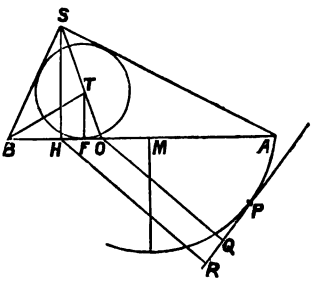
Thus, if a, b, c are as 8, 10, 1, and $m = 20$, then

$$u = 1600x/432, \quad \text{or} \quad 27u = 100x.$$



8870. (Professor NEUBERG.)—Soient S le sommet d'un cône de révolution, et O le point où l'axe du cône rencontre un plan quelconque P. Démontrer que les distances de S et O à une tangente quelconque de la section du cône par le plan P sont dans un rapport constant.

Solution by Professor SCHOUTE.



In the diagram, AB is the major axis of the section, that we suppose elliptical. By Dandelin's theorem the point F, where AB is touched by the in-circle of ASB, is a focus of the section, the plane of which is brought to coincidence with the plane of ASB by rotation about AB. With the notation

$$AM = MB = a, \quad MF = c,$$
$$(a^2-c^2)^{\frac{1}{2}} = b, \quad MO = m,$$

we find $$\frac{SA}{SB} = \frac{AO}{OB} = \frac{a+m}{a-m}, \quad SA - SB = 2c;$$

i.e., $$SA = \frac{c}{m}(a+m), \quad SB = \frac{c}{m}(a-m), \quad SH^2 = \frac{b^2}{m^2}(c^2-m^2).$$

When the eccentric anomaly of P is ϕ, the tangent at P is

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi - 1 = 0,$$

and we find $$OQ = \frac{\frac{m\cos\phi}{a} - 1}{\left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}\right)}, \quad HR = \frac{\frac{c^2\cos\phi}{ma} - 1}{\left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}\right)};$$

therefore, for any value of ϕ, $$\frac{OQ^2}{SH^2 + HR^2} = \frac{m^2}{c^2}.$$

When the section is parabolical or hyperbolical, the proof is similar.

Remarks.—I. In the triangle ASB we find $HM \cdot OM = FM^2$. Therefore we have proved at once the following theorems:—

The line l, the distances u and v of which from two given points O and H verify the relation $pu^2 + qv^2 + r = 0$, envelopes a conic. Of this conic OH is the focal axis, and the foci are separated harmonically by O and H.

The equation contains two parameters, the ratios of the three coefficients p, q, r. Accordingly we can make the conic fulfil two other simple conditions. So we can choose the centre M on OH and the value of the minor axis. And, accordingly, in space we can vary the place of the vertex S of the cone and its angle at the vertex.

II. By the identity $p\alpha^2 + q\beta^2 - (p\alpha^2 + q\beta^2) = 0$, we transform the equation $pu^2 + qv^2 + r = 0$ into

$$p(u^2+\alpha^2) + q(v^2+\beta^2) - (p\alpha^2 + q\beta^2 - r) = 0.$$

Now let O′ and H′ be points in space, the projections of which on the plane of the conic are O and H, and for which $OO' = \alpha$ and $HH' = \beta$. Then $(u^2+\alpha^2)^{\frac{1}{2}} = u'$ and $(v^2+\beta^2)^{\frac{1}{2}} = v'$ are the varying distances of O′ and H′ from the tangents of the conic, and the theorem we have given can be extended as follows:—

"In a given plane the line l, the distances u' and v' of which from two points O′ and H′ given in space verify the relation $pu'^2+qv'^2+r=0$, envelopes a conic. Of this conic OH, &c."

9783. (W. J. C. Sharp, M.A.)—If ABC be a triangle, in which AC > AB, AD the perpendicular from A upon BC, and if DC′ be taken upon CD produced, so that DC′ = CD; prove that the circle through A, B, C′ will also pass through the orthocentre of ABC, and will be equal to the circumcircle of that triangle, which will pass through the orthocentre of ABC′.

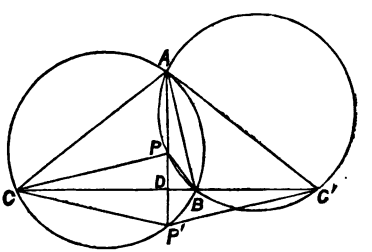
Solution by W. J. Greenstreet, M.A.; G. G. Storr, M.A.; *and others.*

1. Let the circles ABC′ cut AD in P. Then the angle

$APB = \pi - AC'B = \pi - ACB$

$\therefore \quad \angle PAB + PBA = C,$

that is to say, P is the orthocentre of the triangle ABC.



2. If AD meet ACB in P′, we have angle

$AP'C = \frac{1}{2}\pi - P'C'D = \frac{1}{2}\pi - P'CB = \frac{1}{2}\pi - P'AB = ABC = \pi - ABC;$

therefore P′ is the orthocentre of the triangle ABC′.

3. Also $$R = \frac{AB \cdot BC \cdot CA}{4\Delta ABC} = \frac{AB \cdot CA}{2AD} = \frac{AB \cdot BC' \cdot C'A}{2AD \cdot BC'}$$

= circumradius of triangle ABC′.

[This is a part of the geometry of the ambiguous case of triangles. The triangle ABC is determined by the angle C and the sides AB and AC, of which AC is the greater, and the two triangles that can be constructed are ABC and ABC′, whereof ABC′ is the image of the second triangle, as usually obtained in Trigonometry, with respect to the perpendicular from the opposite angle upon the side not given. It appears that, as is well known, the circumcircles are equal, and in the ordinary figure each passes through the image of the orthocentre of the other triangle.]

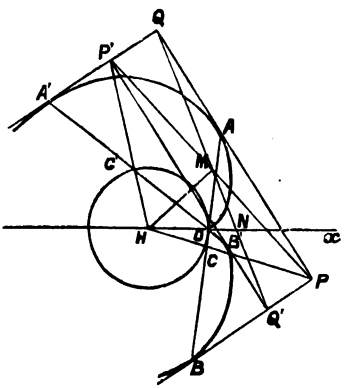
9097. (W. W. Basset, M.A.)—If two tangents be drawn to the cardioid $r = a(1-\cos\theta)$, at the ends of (α) a chord through the cusp, (β) two radii inclined at an angle $\frac{1}{3}\pi$; prove that (1) they will intersect at right angles; and (2) in the first case the locus of their point of intersection is a circle, and in the second case the locus is the limaçon $r = \frac{1}{4}a(2\cos\theta + \sqrt{3})$.

Solution by Professor Schoute.

The cardioid is obtained when on every radius vector OC of the circle (H) through O the segments CA and CB are made equal to the circle's diameter. It has the property that the angle PAO formed by tangent and radius vector is half the angle AOX, which determines the radius vector. This property is immediately verified by the equation

$$r = a(1-\cos\theta),$$

where a is the diameter of (H), and θ the angle XOA.



1. Let P be the point common to the tangents in A and B on the same line through O. Then, as $\angle PAO = \frac{1}{2}\angle AOX$, and in the same way $\angle CBP = \frac{1}{2}\angle XOB$, the sum of the angles A and B of triangle APB is $\frac{1}{2}\pi$; thus P is right angle.

2. It is easily shown that CP passes through the centre H of circle (H). As $\angle AOX = \theta$, we find $\angle PAO = \frac{1}{2}\theta$. But then $\angle PCB = \theta$ and $\angle HCO$ is θ too. So P is situated on HC. And as $CP = a$, the locus of P is the circle concentric with (H), the diameter of which is $3a$.

3. Consider the quadrangle AOA'Q on the supposition that $\angle A'OA$ is $\frac{1}{3}\pi$; then

$$\angle AOX = \theta, \quad \angle A'OX = \theta + \tfrac{1}{3}\pi,$$
$$\angle PAO = \tfrac{1}{2}\theta, \quad \angle QA'O = \tfrac{1}{2}(\theta + \tfrac{1}{3}\pi).$$

The sum of the angles in A, O, A' is

$$(\pi - \tfrac{1}{2}\theta) + \tfrac{1}{3}\pi + \tfrac{1}{2}(\theta + \tfrac{1}{3}\pi) = \tfrac{5}{6}\pi,$$

and the angle in Q must be a right angle.

4. We consider the tangents in the four points A, B, A', B' situated on the two lines OC and OC' through O. They give rise to the points P and P', Q and Q'. As $\angle A'OA = \angle B'OB = \frac{1}{3}\pi$ and $\angle AOB = \angle A'OB' = \pi$, the quadrangle P'QP'Q' is a rectangle. The diagonals PP' and QQ' of this rectangle are constant. For, as $HP = HP' = \frac{1}{2}a$ and $\angle P'HP = \frac{2}{3}\pi$, we find $PP' = \frac{1}{2}a\sqrt{3}$. Moreover, the mid-point M of PP' describes the circle concentric to (H), the diameter of which is equal to $\frac{1}{2}a$. So it is evident that Q and Q' will describe a limaçon when QQ' passes through a fixed point. For the chords Q, Q' have a constant length, and the mid-

point describes a circle. Now the fixed point in question is the point N on OX. For we find easily HN = HM. Indeed we have

$$\angle MHN = \angle MHP - \angle OHP = \tfrac{1}{2}\pi - (\pi - 2\theta) = 2(\theta - \tfrac{1}{2}\pi),$$

$$\angle NMH = \angle PMH - \angle PMQ' = \tfrac{1}{2}\pi - 2\angle PP'Q'$$

$$= \tfrac{1}{2}\pi - 2\angle(HM, BP) = \tfrac{1}{2}\pi - 2(\angle MHC - \angle BPC)$$

$$= \tfrac{1}{2}\pi - 2(\tfrac{1}{2}\pi - \angle CBP) = \tfrac{3}{2}\pi + \theta.$$

Therefore $\angle NMH = \angle HNM$, &c.

10193. (Professor Moreau.)—Trouver le lieu des centres des cercles qui passent par le sommet A d'un triangle donné et coupent les côtés AB, AC respectivement aux points B′, C′ tels que BB′ = CC′.

Solution by E. M. Langley, M.A.; D. T. Griffiths; *and others.*

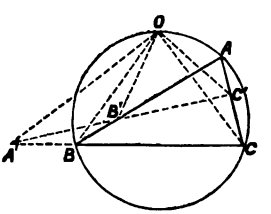
Let O be the mid-point of the arc BAC of the circumcircle of BAC; then

$$OB, BB' = OC, CC',$$

$$\angle OBB' = OCC',$$

$$\therefore \quad \angle BOB' = COC',$$

$$\therefore \quad \angle B'OC' = BOC = B'AC';$$



hence the circumcircle of B′AC′ passes through O; and thus the locus of its centre is the perpendicular bisector of AO. The solution is easily adapted to the case BB′ : CC′ = m : n.

Again, since circumcircles of BA′B′, CA′C′ are known also to pass through O,

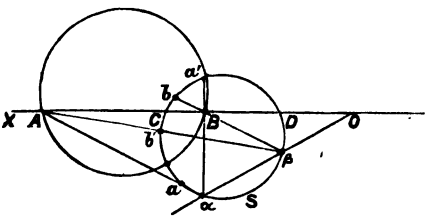
$$\angle A'OB' = ABC, \quad \angle OA'B' = OBA, \text{ and } \angle OC'A' = OCB.$$

OA′C′ is of constant species, and A′B′C′ divided in constant ratio. Hence A′B′C′ is a limiting case of a triangle of constant species inscribed in ABC, and is therefore a special case of 10056. Also, since *any* point P cutting B′C′ in constant ratio describes a straight line, the above affords an easy demonstration of Newton's *Lemma* xxiii. (Book 1, § 5.)

8995. (J. L. McKenzie, B.A., B.Sc.)—A system of coaxal circles, of which one circle S is considered fixed, is cut by a fixed line OX. If A, B be the points in which OX cuts any circle of the system, and if any line through O cut the fixed circle S in α and β; prove that Aα, Bβ cut the circle S again in points which are collinear with a fixed point on OX.

Solution by Professor SCHOUTE.

The coaxal circles cut OX in the quadratic involution determined by the couples AB and CD (where C, D are the points common to S and OX). Project the two homographic divisions



$$\begin{cases} A, B\,;\ C, D\,;\ E, F;\ G, H\,;\ \ldots \\ B, A\,;\ D, C\,;\ F, E\,;\ H, G\,;\ \ldots \end{cases}$$

which, combined, form this involution, the first from a, the second from β, on S. Then we obtain the homographic divisions

$$\begin{cases} a, a'\,;\ C, D\,;\ \ldots \\ b, b'\,;\ D, C\,;\ \ldots \end{cases}$$

But these homographic divisions on S are in involution, C and D corresponding with one another. Then the lines ab, $a'b'$, CD ..., joining corresponding points, are concurrent, &c.

The point common to ab, $a'b'$, CD ... being conjugate to O in the involution (AB, CD), it will not change when $O\alpha\beta$ turns about O and another circle of the system takes the place of S.

10238. (J. J. BARNIVILLE.)—A triangle, in which $a^2 : b^2 : c^2 = 9 : 7 : 16$, performs a complete rotation round the side a; prove that it generates a cone whose vertical angle is equal to the trihedral angle of a cube.

Solution by H. J. WOODALL; Rev. T. GALLIERS, M.A.; *and others.*

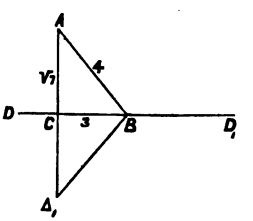
Let $AD\Delta_1D_1$ be section of sphere; therefore $AC\Delta_1B$ is section of cone;

$DC = AB - CB = 4 - 3 = 1$ and $CD_1 = 7$,

therefore surface of zone from pole D to small circle $ACA_1 = \frac{1}{8} \times$ surface of sphere, therefore solid angle of cone

$= \frac{1}{8}$ whole solid angle at any point

= solid angle of cube.



10373. (W. J. GREENSTREET, M.A.)—Through C the mid-point of the arc ACB of a circle, any chord CE cuts the straight line AB in D; find the locus of the centre of the circle through B, D, E.

Solution by R. KNOWLES, B.A.; G. G. STORR, M.A.; *and others.*

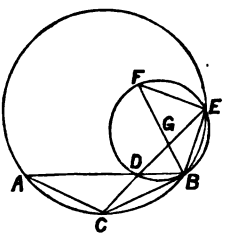
Join CB, BG, G being the centre of the circle through B, D, and E; produce BG to F, and join EF and EB; then

$$\angle BAC = BEC = ABC;$$

also $\angle FED = FBD$;

therefore $\angle FEB = FBC$;

but FEB is a right angle, being in a semicircle, hence FBC is a right angle; thus CB is fixed, and therefore FB; therefore centre lies on FB perpendicular to CB.



If the points A and B are interchanged by a similar proof, it can be proved that the centre lies on the straight line drawn from the extremity of the chord at right angles to BC.

Again, if the straight line CE be drawn on the other side of the diameter through C, it can be proved in a similar way that the centre lies on the same two fixed lines, and there meet at the extremity of the diameter through C.

10385. (Professor GOB.)—Soit ABCD un parallélogramme quelconque, et soient A', B', C' les seconds points de rencontre des droits DA, DB, DC avec la circonférence circonscrite au triangle ABC. Démontrer que la droite A'D est la moyenne géométrique, en grandeur et en position, des droites A'C' et A'B'.

Solution by D. BIDDLE; R. KNOWLES, B.A.; *and others.*

There are three cases under this question, viz.: (1) where B', C' are both *further* from D than B, C; (2) where both are *nearer* than B, C; (3) where one is nearer and the other further. But the same reasoning can be adapted to all. Thus,

$\angle A'C'C = A'AC$ (or its supplement),

$\angle A'B'B = A'AB$ (or its supplement).

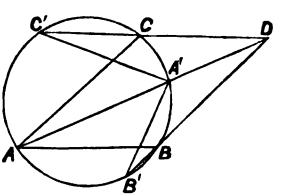


Fig. 1.

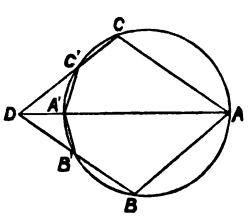


Fig. 2.

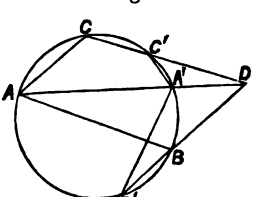


Fig. 3.

Therefore $\angle A'C'D = A'AC$, and $\angle A'B'D = A'AB$.

Hence A′C′D, ACD are similar triangles; likewise A′B′D, ABD; and of these, ACD, ABD are already known to be similar each to each. Consequently A′C′D, A′B′D are similar, having the $\angle DA'C' = DA'B$, and the sides $DA' : A'C' = A'B' : DA'$.

10338. (C. Bickerdike.)—If the density at any point of a triangular lamina varies as the product of the perpendiculars on the sides, prove that the mean density is $\frac{9}{20}$ of the density at the centres of inertia of the area of the triangle.

Solution by Rev. T. R. Terry, M.A.; G. G. Storr, M.A.; *and others.*

Take the sides CB and CA as axes of x and y. Then, if μ be the density at the centre of mass, the density D at the point (x, y) is

$$27\mu \frac{x}{a} \cdot \frac{y}{b} \left(1 - \frac{x}{a} - \frac{y}{b}\right).$$

Therefore the mean density $= \frac{2}{ab \sin C} \iint D\, dx\, dy \sin C$;

and, integrating for y from 0 to $b(1 - x/a)$, and for x from 0 to a, we get the required result.

10284. (J. J. Barniville.)—If C_x^y denote the combinations of y things taken x at a time, find C_x^{y+z} in terms of C_x^y and C_x^z.

Solution by W. J. Greenstreet, M.A.; Prof. Beyens; *and others.*

We have the following classification of C_x^{y+z}:—

(1) C_x^y; (2) each of the C_{x-1}^y with each of the C_1^z; *i.e.*, $C_{x-1}^y C_1^z$;

(3) each of the C_{x-2}^y with each of the C_2^z,

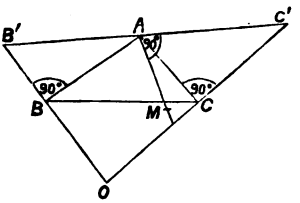
and so on, a classification exhaustive and simplex. Therefore

$$C_x^{y+z} = C_x^y + C_{x-1}^y C_1^z + C_{x-2}^y C_2^z + \ldots + C_1^y C_{x-1}^z + C_x^z.$$

10361. (Maurice d'Ocagne.)—Soit M un point pris sur la base BC du triangle ABC. Si la perpendiculaire élevée en A à la droite AM coupe en B′ et en C′ les perpendiculaires élevées respectivement à AB et à AC,

en B et en C ; démontrer que (1) AB′. MB/AC′. MC = AB²/AC² ; et (2) si AM est symédiane de ABC, le sommet A est le milieu de B′C′.

Solution by W. J. Greenstreet, M.A. ; R. Whapham, B.A. ; *and others.*



AB′ = AB cosec BAM,
AC′ = AC cosec CAM,
MB = AB sin BAM cosec AMB,
MC = AC sin CAM cosec AMC ;

therefore $\frac{AB'.MB}{AC'.MC} = \frac{AB^2}{AC^2}$;

and, if AM be a symmedian,

$$MB : MC = AB^2 : AC^2 ;$$

therefore AB′ = AC′ ; *i.e.*, A is mid-point of B′C′.

10315. (Professor Cesáro.)—Démontrer l'identité

$$1 - C_{2n,1}^2 + C_{2n,2}^2 + C_{2n,3}^2 + \ldots + 1 = (-1)^n C_{2n,n}.$$

Solution by Rev. T. R. Terry, M.A. ; Rev. J. L. Kitchin ; *and others.*

Since $(1-x)^{2n} = 1 - C_{2n,1}\,x + C_{2n,2}\,x^2 - \ldots,$

and $(1+x)^{2n} = x^{2n} + C_{2n,1}\,x^{2n-1} + \ldots,$

therefore the series = coefficient of x^{2n} in $(1-x^2)^{2n}$

= coefficient of y^n in $(1-y^{2n}) = (-1)^n C_{2n,n}$.

10324. (R. H. W. Whapham, B.A.)—If a parabola be drawn having contact of the third order with a conic, prove that the axis of the parabola is parallel to the line joining the point of contact to the centre of the conic.

Solution by Rev. T. R. Terry, M.A. ; Professor Schoute ; *and others.*

The equation to any conic touching the axis of x at the origin is (1) $y = Ax^2 + 2Bxy + Cy^2$; and, differentiating, we see that, at the origin, $y_1 = 0$, $y_2 = 2A$, $y_3 = 12AB$; hence, in conics having contact of the third order, C only can vary.

If C be chosen so that (1) represents a parabola, the axis is parallel to $Ax + By = 0$. But this is clearly the line joining the origin to the centre of (1), therefore, &c.

[The conics that pass through the four points A, B, C, D, situated on the parallels AD, BC, have the line EF that joins the mid-points E, F of AD, BC in common as diameter, &c.]

10224. (Professor DE LONGCHAMPS.)—Résoudre les équations

$$\frac{x-b-c}{x-2a}+\frac{x-c-a}{x-2b}+\frac{x-a-b}{x-2c}+1=0 \quad \ldots\ldots\ldots\ldots (1),$$

$$\frac{2x+c-b}{x+b-c}+\frac{2x+a-c}{x+c-a}+\frac{2x+b-a}{x+a-b}+2=0 \quad \ldots\ldots\ldots\ldots (2),$$

$$\frac{x+b+c-a}{x+3a-b-c}+\frac{x+c+a-b}{x+3b-c-a}+\frac{x+a+b-c}{x+3c-a-b}+1=0 \quad \ldots\ldots\ldots (3);$$

et reconnaître que la *clef* qui peut servir à former, en nombre indéfini, les équations de ce genre est représentée par l'identité

$$(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)-\alpha\beta(\alpha+\beta)-\beta\gamma(\beta+\gamma)-\gamma\alpha(\gamma+\alpha)-2\alpha\beta\gamma=0.$$

Solution by J. BARNIVILLE; H. J. WOODALL; *and others.*

In (1), making $\alpha=x-2a$, $\beta=x-2b$, $\gamma=x-2c$, we have

$$\alpha\beta(\alpha+\beta)+\beta\gamma(\beta+\gamma)+\gamma\alpha(\gamma+\alpha)+2\alpha\beta\gamma=0;$$

therefore, according to the "clef,"

$$(x-a-b)(x-b-c)(x-c-a)=0,$$

and the roots are $a+b$, $b+c$, $c+a$.

In (2), α represents $x+b-c$; β, $x+c-a$; and γ, $x+a-b$. Hence

$$(2x+b-a)(2x+a-c)(2x+c-b)=0,$$

and the roots are $\frac{1}{2}(a-b)$, $\frac{1}{2}(b-c)$, $\frac{1}{2}(c-a)$.

In (3) we find, in like manner, that

$$(x+a+b-c)(x+b+c-a)(x+c+a-b)=0,$$

the roots being $a-b-c$, $b-c-a$, $c-a-b$.

3944. (J. J. WALKER, F.R.S.)—If AD is the arc drawn from one angle A of a spherical triangle so as to bisect that angle and meet the opposite side BC in D, and if the arcs bisecting the other angles meet AD in O, prove that

$$\frac{\tan AD}{\tan OA}=\frac{2\sin s\cos(s-a)}{\sin(b+c)}.$$

Solution by W. J. C. SHARP, M.A.; W. J. GREENSTREET, M.A.; *and others.*

Applying to the triangles AOB, ADB the formula

$$\cot a\sin c=\cot A\sin B+\cos c\cos B,$$

we have $\cot AO \sin c = \cot \frac{1}{2}B \sin \frac{1}{2}A + \cos c \cos \frac{1}{2}A$,

$\cot AD \sin c = \cot B \sin \frac{1}{2}A + \cos c \cos \frac{1}{2}A$,

therefore $$\frac{\tan AD}{\tan AO} = \frac{\cot \frac{1}{2}B \sin \frac{1}{2}A + \cos c \cos \frac{1}{2}A}{\frac{1}{2}(\cot \frac{1}{2}B - \tan \frac{1}{2}B) \sin \frac{1}{2}A + \cos c \cos \frac{1}{2}A}$$

$$= \frac{2 \sin s \sin c \cos (s-a)}{\sin s \sin c \cos (s-a) + \sin (s-a) \sin c \cos s} = \frac{2 \sin s \sin (s-a)}{\sin (b+c)}.$$

[Tan AD $= 2 \cos \frac{1}{2}A / (\cot b + \cot c) = 2 \cos \frac{1}{2}A \sin b \sin c / \sin (b+c)$,

and $\tan AO = \tan (s-a) / \cos \frac{1}{2}A$;

thus, if Mr. WALKER's equation holds, we must have

$$\cos^2 \tfrac{1}{2}A \sin b \sin c = \sin s \cdot \sin (s-a),$$

which is seen to be true by simple substitution for $\cos^2 \frac{1}{2}A$.]

10353. (The EDITOR.)—Show that, if $s = \frac{1}{2}(a+b+c)$, $s_1 = s-a$, &c., the solution of the equations

$$\frac{x+y}{a} = \frac{y+z}{b} = \frac{z+x}{c}, \quad mxy + nyz + pzx = xyz$$

is given by $$z = s_1 \left(\frac{m}{s_1} + \frac{n}{s_2} + \frac{p}{s_3} \right), \text{ &c.}$$

Solution by Prof. CURTIS; W. J. GREENSTREET, M.A.; *and others.*

Let $$k = \frac{x+y}{a} = \frac{y+z}{b} = \frac{z+x}{c} = \frac{x+y+z}{s};$$

then $k = \frac{z}{s_1} = \frac{x}{s_2} = \frac{y}{s_3}$; $\therefore k = \frac{m}{s_1} + \frac{n}{s_2} + \frac{p}{s_3}$; $\therefore z = k s_1 = \text{&c.}$

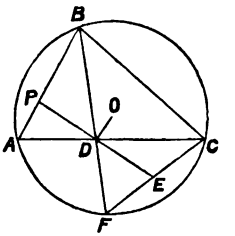
10265. (Professor SYAMADAS MUKHOPADHYAY, B.A.) — A hexagon, A, B, C, D, E, F, has three pairs of opposite sides parallel. Prove, by elementary geometry, that the triangles ACE and BDF are of equal area, and that the three lines joining middle points of opposite sides have a common point.

Solution by J. J. BARNIVILLE; J. C. HOROBIN, B.A.; *and others.*

AF = DC, FE = CB, EA = BD, therefore the triangle AEF is equal in every respect to BCD; in like manner ECD = ABF, CBA = DEF, therefore AEF + CED + ABC = ABF + BCD + DEF. Subtracting both sides from the entire hexagon, we have ACE = BDF. Again, since the three diagonals of the hexagon bisect each other, the three parallelograms which it contains have a common centre; hence the lines joining $\frac{1}{2}$-points of opposite sides are concurrent.

10325. (D. T. Griffiths.)—O is the circumcentre of the triangle ABC, and P a variable point in AB (or in AB produced). The circle on OP as diameter meets AC in D, and PD is produced to E, so that DE = PD. Show that the locus of intersection of BD, CE is the circumcircle ABC.

Solution by Professors Schoute, Nilkantha Sarkar, *and others.*

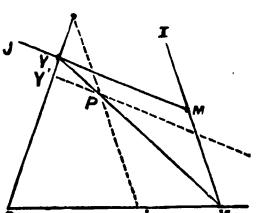


Let F be an arbitrary point of the circumcircle, D the intersection of AC and BF. Any line through D is met by the conics passing through A, B, C, F in an involution of which D is a double point. If D is the mid-point of PE, the other double point is at infinity; then D is also the mid-point of the segment determined on PE by the circumcircle, hence the angle ODE = 90°, and the circle on OP as diameter meets AC in D.

[The envelope of PD is the parabola of which O is the focus and AC the tangent at the vertex, and the locus of E is a conic through A and C that touches AB in A.]

10283. (Emile Vigarié.)—Etant donnés un angle YOX et un point P dans son plan, on mène par ce point une droite quelconque YPX qui coupe OY en Y et OX en X. Par X et Y on trace deux droites XM, YM respectivement parallèles à deux directions données. Trouver le lieu du point M quand XY varie.

Solution by J. C. St. Clair; R. Knowles, B.A.; *and others.*



Let I, J be the points at infinity in the given directions. Then, since X, Y form homographic ranges, the rays IX, JY form homographic pencils, and their intersections lie on a conic through I, J, *i.e.*, an hyperbola of which the asymptotes are parallel to the given directions. This hyperbola evidently passes through O; and if through P we draw PX', PY' parallel to the directions I, J, they will meet OY, OX in points which belong to the locus; thus the hyperbola is determined by three finite points and two points at infinity.

10077. (Professor Wolstenholme, Sc.D.)—The signs of the terms of a certain series are determined by the following rule: if the first 2^r terms be taken, the signs of the second half of these terms are respectively

opposite to the corresponding signs of the first half; find the sign of the n^{th} term, that of the first being positive.

Solution by H. J. Woodall; Prof. Mukhopâdhyây; *and others.*

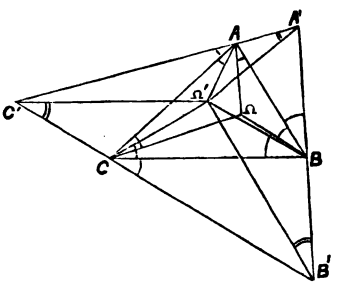
We can find m so that $n = 2^m + k$, 2^m being the highest power of 2 less than n; then sign of u_n = − sign of k, if similarly $k = 2^n + j$; sign of u_n = − sign of k = −(− sign of j) = + sign of j, and so on. There are two cases:—

1. If n is odd, the last remainder is 1. Then, expressing n in the scale of notation whose radix is 2, we shall obtain s significant figures; then, since each significant figure corresponds to a certain power of 2 which is subtracted from n, therefore the sign of $u_n = (-)^{s-1}$.

2. If n is even. Expressing n in the scale of notation whose radix is 2, we shall obtain the t significant figures, and the last figure is r places (inclusive) from the "units" place. In this case the sign of $u_n = (-)^{s-1}$, the sign of the last significant figure $= (-)^{(s-1)+r-1} = (-)^{s+r}$.

10266. (Professor Gob.)—Soient Ω et Ω' les points de Brocard du triangle ABC ($\Omega AB = \Omega' AC = \omega$). En menant par B, C, A des parallèles respectivement à ΩA, ΩB, ΩC, on forme un nouveau triangle A′B′C′. Démontrer que Ω est le point de L moine, et Ω' un des points de Brocard de A′B′C′.

Solution by Professor Schoute.



The triangles ABC, A′B′C′ are similar, and the angles denoted by a single arc are equal to ω by definition.

Now the triangles $A\Omega'B$ and $AA'B$ give

$$A\Omega' : AB = \sin\omega : \sin A,$$

$$AA' : AB = \sin\omega : \sin A;$$

therefore $A\Omega' = AA'$, and in the same manner $B\Omega' = BB'$ and $C\Omega' = CC'$. So $A'\Omega'$ is parallel to the bissectrix AC of the angle $\Omega'AC'$, &c., and the angles denoted by two arcs are equal to ω. Therefore Ω' is a Brocard-point for A′B′C′.

For the distances a', b', c' of Ω to B′C′, C′A′, A′B′, we find

$$a' = C\Omega \sin C, \quad b' = A\Omega \sin A, \quad c' = B\Omega \sin B.$$

But triangle $C\Omega B$ gives $C\Omega \sin C = CB \sin\omega$, &c.; so we find

$$a' : b' : c' = CB : AC : BA = B'C' : C'A' : A'B',$$

which proves Ω to be the Lemoine-point for A′B′C′.

The double point of the similar figures ABC and A′B′C′ is Ω'.

10271. (Capitaine DE ROCQUIGNY.)—Déterminer n de manière que $1^3+3^3+5^3+\ldots+(2n-1)^3$ soit égale à un carré.

Solution by J. C. ST. CLAIR; J. J. BARNIVILLE; *and others.*

Since $\Sigma n^3 = \{\frac{1}{2}n(n+1)\}^2$,

sum of all cubes to $(2n-1)^3 = n^2(2n-1)^2$,

sum of even cubes to $(2n-2)^3 = 2n^2(n-1)^2$;

hence sum of odd cubes $= n^2\left[(2n-1)^2-2(n-1)^2\right] = n^2(2n^2-1)$.

Therefore $2n^2-1$ is a square by supposition, and if we put it $= y^2$, the values of n and y are evidently the numerators and denominators of the alternate convergents to $1/\sqrt{2}$; whence $n = 1, 5, 29$; and generally

$$n = \tfrac{1}{4}\sqrt{2}\left\{(1+\sqrt{2})^m-(1-\sqrt{2})^m\right\},$$

where m is any odd integer.

9776. (W. J. C. SHARP, M.A.)—If perpendiculars p, q, r be drawn from the vertices of a triangle upon any tangent to the circumcircle, prove that these are connected by the relation

$$\begin{vmatrix} 0, & p, & q, & r \\ p, & 0, & c^2, & b^2 \\ q, & c^2, & 0, & a^2 \\ r, & b^2, & a^2, & 0 \end{vmatrix} = 0,$$

and that a similar relation holds in space of n dimensions.

Solution by Professor SEBASTIAN SIRCOM, M.A.

In areal coordinates the equations of the circumcircle and of a line are

$$a^2\mu\nu + b^2\nu\lambda + c^2\lambda\mu = 0, \quad p\lambda + q\mu + r\nu = 0;$$

that this may be a tangent at λ_1, μ_1, ν_1, we must have

$$c^2\mu_1 + b^2\nu_1 = mp, \quad c^2\lambda_1 + a^2\nu_1 = mq,$$
$$b^2\lambda_1 + a^2\mu_1 = mr, \quad p\lambda_1 + q\mu_1 + r\nu_1 = 0;$$

eliminating λ_1, μ_1, ν_1, we obtain the required relation.

Similarly, in content-coordinates the equation of the circumspheric is

$$(1, 2)^2\lambda\mu + (2, 3)^2\mu\nu + (3, 1)^2\nu\lambda + \ldots = 0,$$

and the equation of a linear locus $p\lambda + q\mu + r\nu + \ldots = 0$; that this may be a tangent at $\lambda_1, \mu_1, \nu_1 \ldots$ we must have $(1, 2)^2\mu_1 + (1, 3)^2\nu_1 + \ldots = mp$, and $n-1$ others; eliminating $\lambda_1, \mu_1, \nu_1 \ldots$ from these, and $p\lambda_1 + q\mu_1 + r\nu_1 + \ldots = 0$, we obtain the required relation.

[If p, q, r, &c., p', q', r', &c., be the perpendiculars from the vertices of a simplicissimum in space of n dimensions upon parallel linear tangent loci to the circumspheric $p'-p = q'-q = \ldots = 2R$, and by the given

equation
$$R\begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ 1, & 0, & (1.2)^2, & (1.3)^2, & \dots \\ 1, & (1.2)^2, & 0, & (2.3)^2, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$
$$= -\begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ p, & 0, & (1.2)^2, & (1.3)^2, & \dots \\ q, & (1.2)^2, & 0, & (2.3)^2, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix},$$

and if
$$(1.2)^2\mu + (1.3)^2\nu + \dots = 0$$
be the first locus, so that $p = 0$, $q = (1.2)^2 k$, $r = (1.3)^2 k$, &c., by a formula proved, *Proc. Lond. Math. Soc.*, Vol. XVIII., p. 353, and identical with Professor Sylvester's Question 8078,

$$k^2\begin{vmatrix} 0, & 0, & 0, & (1.2)^2, & (1.3)^2, & \dots \\ 0, & 0, & 1, & 1, & 1, & \dots \\ 0, & 1, & 0, & (1.2)^2, & (1.3)^2, & \dots \\ (1.2)^2, & 1, & (1.2)^2, & 0, & (2.3)^2, & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$
$$\equiv k^2\begin{vmatrix} 0, & -1, & 0, & 0, & 0, & \dots \\ -1, & 0, & 1, & 1, & 1, & \dots \\ 0, & 1, & 0, & (1.2)^2, & (1.3)^2, & \dots \\ 0, & 1, & (1.2)^2, & 0, & (2.3)^2, & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}$$
$$\equiv -k^2\begin{vmatrix} 0, & (1.2)^2, & (1.3)^2, & \dots \\ (1.2)^2, & 0, & (2.3)^2, & \dots \\ (1.3)^2, & (2.3)^2, & 0, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = (-2)^{n-1}(n!)^2 V^2;$$

$$\therefore\ R^2(-2)^{2n}(n!)^4 V^4 = k^2\begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ 0, & 0, & (1.2)^2, & (1.3)^2, & \dots \\ (1.2)^2, & (1.2)^2, & 0, & (2.3)^2, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}^2$$
$$= k^2\begin{vmatrix} 0, & (1.2)^2, & (1.3)^2, & \dots \\ (1.2)^2, & 0, & (2.3)^2, & \dots \\ (1.3)^2, & (2.3)^2, & 0, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}^2$$
$$= -(-2)^{n-1}(n!)^2 V^2\begin{vmatrix} 0, & (1.2)^2, & (1.3)^2, & \dots \\ (1.2)^2, & 0, & (2.3)^2, & \dots \\ (1.3)^2, & (2.3)^2, & 0, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix};$$

$$\therefore\ \begin{vmatrix} 0, & (1.2)^2, & (1.3)^2, & \dots \\ (1.2)^2, & 0, & (2.3)^2, & \dots \\ (1.3)^2, & (2.3)^2, & 0, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = -(-2)^{n+1}(n!)^2 V^2 R^2$$

(Question 8976.).]

2827. (Professor SYLVESTER.)—If four ternary quadrics in ϕ, ψ, ω, which all vanish for one and the same set of values of the variables, be respectively equal to x, y, z, t, &c.; prove, by a simple application of the dialytic method of elimination, that these last-named four quantities are connected by a cubic equation, and consequently that every cubic surface is unicursal.

Solution by Professor SCHOUTE.

As
$$x = F_1(\phi, \psi, \omega), \quad y = F_2(\phi, \psi, \omega),$$
$$z = F_3(\phi, \psi, \omega), \quad t = F_4(\phi, \psi, \omega)$$

vanish for the same set of values of ϕ, ψ, ω, the conics $F_1 = 0$, $F_2 = 0$, $F_3 = 0$, $F_4 = 0$ have a common point P. By a linear transformation of ϕ, ψ, ω in p, q, r, with reference to a trilinear system for which P is the vertex $q = 0$, $r = 0$ of the triangle of coordinates, we find

$$\left.\begin{aligned} x &= b_1q^2 + c_1r^2 + f_1qr + g_1rp + h_1pq \\ y &= b_2q^2 + c_2r^2 + f_2qr + g_2rp + h_2pq \\ z &= b_3q^2 + c_3r^2 + f_3qr + g_3rp + h_3pq \\ t &= b_4q^2 + c_4r^2 + f_4qr + g_4rp + h_4pq \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

When we multiply these four equations successively by p, q, r, we obtain 12 equations homogeneous in the 12 quantities

$$p,\ q,\ r,\ p^2q,\ pq^2,\ q^3,\ p^2r,\ pr^2,\ r^3,\ q^2r,\ qr^2,\ pqr,$$

i.e., in the quantities of the first and of the third order in p, q, r except p^3. By eliminating these quantities in the indicated order, we find the annexed determinant equated to zero as the equation, which is that of a cubic surface. As the equations (1) show, this equation contains 20 constants, and is, therefore, the most general of its order. So the coordinates of every surface of the third order can be represented by the equations (1); in other words, every surface of the third order is unicursal.

$$\begin{vmatrix}
x & 0 & 0 & h_1 & b_1 & 0 & g_1 & c_1 & 0 & 0 & 0 & f_1 \\
0 & x & 0 & 0 & h_1 & 0 & 0 & 0 & 0 & f_1 & c_1 & g_1 \\
0 & 0 & x & 0 & 0 & b_1 & 0 & g_1 & c_1 & b_1 & f_1 & h_1 \\
y & 0 & 0 & h_2 & b_2 & 0 & g_2 & c_2 & 0 & 0 & 0 & f_2 \\
0 & y & 0 & 0 & h_2 & 0 & 0 & 0 & 0 & f_2 & c_2 & g_2 \\
0 & 0 & y & 0 & 0 & b_2 & 0 & g_2 & c_2 & b_2 & f_2 & h_2 \\
z & 0 & 0 & h_3 & b_3 & 0 & g_3 & c_3 & 0 & 0 & 0 & f_3 \\
0 & z & 0 & 0 & h_3 & 0 & 0 & 0 & 0 & f_3 & c_3 & g_3 \\
0 & 0 & z & 0 & 0 & b_3 & 0 & g_3 & c_3 & b_3 & f_3 & h_3 \\
t & 0 & 0 & h_4 & b_4 & 0 & g_4 & c_4 & 0 & 0 & 0 & f_4 \\
0 & t & 0 & 0 & h_4 & 0 & 0 & 0 & 0 & f_4 & c_4 & g_4 \\
0 & 0 & t & 0 & 0 & b_4 & 0 & g_4 & c_4 & b_4 & f_4 & h_4
\end{vmatrix}$$

8457. (W. J. C. SHARP, M.A.)—If E be the spherical excess of a triangle whose sides are a, b, c, show (1) that

$\sin^2 \frac{1}{4}E = \sin \frac{1}{2}s \sin \frac{1}{2}(s-a) \sin \frac{1}{2}(s-b) \sin \frac{1}{2}(s-c) / \cos \frac{1}{2}a \cos \frac{1}{2}b \cos \frac{1}{2}c$,

where $2s = a + b + c$; and hence (2) deduce the ordinary formula for the area of a plane triangle.

Solution by Professor LAMPE.

1. Mr. SHARP seems to have in view a method commonly employed in German text-books, *e.g*, in the *Lehrbuch der sphärischen Trigonometrie*, by SPITZ, Leipzig, 1886, where the required formulæ are deduced from properties of the rectangular triangle. Starting from the Gaussian equations,

$$(1)\quad \frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}=\frac{\cos\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}\gamma},\quad \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}=\frac{\sin\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}\gamma},$$

we have $\cos\frac{1}{2}\mathrm{E}\cos\frac{1}{2}c=\cos\frac{1}{2}a\cos\frac{1}{2}b+\sin\frac{1}{2}a\sin\frac{1}{2}b\cos\gamma.$

Now $$\cos\gamma=\frac{\cos^2\frac{1}{2}a+\cos^2\frac{1}{2}b+\cos^2\frac{1}{2}c-1-2\cos^2\frac{1}{2}a\cos^2\frac{1}{2}b}{2\sin\frac{1}{2}a\cos\frac{1}{2}a\sin\frac{1}{2}b\cos\frac{1}{2}b},$$

$$\therefore\quad (2)\ \cos\tfrac{1}{2}\mathrm{E}=\frac{\cos^2\frac{1}{2}a+\cos^2\frac{1}{2}b+\cos^2\frac{1}{2}c-1}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}\ \text{(p. 82 of SPITZ)}.$$

Forming $\frac{1}{2}(1-\cos\frac{1}{2}\mathrm{E})$ and $\frac{1}{2}(1+\cos\frac{1}{2}\mathrm{E})$, we get

$$(3)\quad \sin^2\tfrac{1}{2}\mathrm{E}=\frac{\sin\frac{1}{2}s\sin\frac{1}{2}(s-a)\sin\frac{1}{2}(s-b)\sin\frac{1}{2}(s-c)}{\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c},$$

and $$(4)\quad \cos^2\tfrac{1}{2}\mathrm{E}=\frac{\cos\frac{1}{2}s\cos\frac{1}{2}(s-a)\cos\frac{1}{2}(s-b)\cos\frac{1}{2}(s-c)}{\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}.$$

Dividing (3) by (4), we obtain LHUILIER'S formula,

$$(5)\quad \tan^2\tfrac{1}{2}\mathrm{E}=\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c).$$

Developing $\cos\frac{1}{2}(a+\beta+\gamma)=-\sin\frac{1}{2}\mathrm{E}$ in the same way from (1), we get

$$\sin\tfrac{1}{2}\mathrm{E}=(2\sin\tfrac{1}{2}\gamma\cos\tfrac{1}{2}\gamma\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b)/\cos\tfrac{1}{2}c,$$

or, replacing $\sin\frac{1}{2}\gamma$ and $\cos\frac{1}{2}\gamma$ by their expressions in a, b, c,

$$(6)\quad \sin\tfrac{1}{2}\mathrm{E}=\left\{\sin s\sin(s-a)\sin(s-b)\sin(s-c)\right\}^{\frac{1}{2}}/(2\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c).$$

[2. This part of the Question, which Professor LAMPE does not solve, may be deduced as follows:—

If r be the radius of the sphere, the formula may be written

$$\left\{\frac{\mathrm{E}}{4r}-\frac{\mathrm{E}^3}{3!\,(4r)^3}+\dots\right\}^2=$$

$$\frac{\left(\frac{s}{2r}-\frac{1}{3!}\frac{s^3}{(2r)^3}+\dots\right)\dots\left(\frac{s-a}{2r}-\frac{1}{3!}\frac{(s-b)^3}{2r^3}\right)\left(\frac{s-c}{2r}-\&c.\right)}{\left(1-\frac{a^2}{2!\,(2r)^2}+\dots\right)\left(1-\frac{b^2}{2!\,(2r)^2}+\dots\right)\left(1-\frac{c^2}{2!\,(2r)^2}+\dots\right)};$$

or, if r become very great, $$\frac{\mathrm{E}^2}{16r^2}=\frac{s(s-a)(s-b)(s-c)}{16r^4},$$

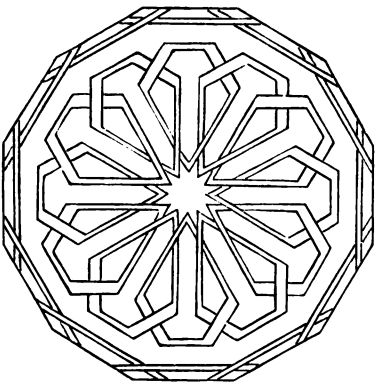
or the well-known formula, $(\text{area})^2=r^2\mathrm{E}^2=s(s-a)(s-b)(s-c)$.

If E' be the spherical excess of a triangle each of the sides of which are half the corresponding sides of the original triangle, by CAGNOLI'S theorem, $$\sin^2\tfrac{1}{2}\mathrm{E}'=\frac{\sin\frac{1}{2}s\sin\frac{1}{2}(s-a)\sin\frac{1}{2}(s-b)\sin\frac{1}{2}(s-c)}{4\cos^2\frac{1}{4}a\cos^2\frac{1}{4}b\cos^2\frac{1}{4}c},$$

and $\sin^2\frac{1}{2}\mathrm{E}':\sin^2\frac{1}{2}\mathrm{E}=\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c:4\cos^2\frac{1}{4}a\cos^2\frac{1}{4}b\cos^2\frac{1}{4}c.$

For another solution, see Vol. XLVIII., p. 96.]

10269. (F. R. J. Hervey.)—Cut out the accompanying polygon (erasing the black outline), and cut it into twelve parts, from which reconstruct the polygon, turning the whole figure into its perversion, or image in a plane mirror. [This is one of many species of symmetrical transformation obtainable by the same process, which is applicable to any regular polygon having an even number of sides; the effect produced depending upon the conditions observed in drawing the original figure.]



Solution by the Proposer.

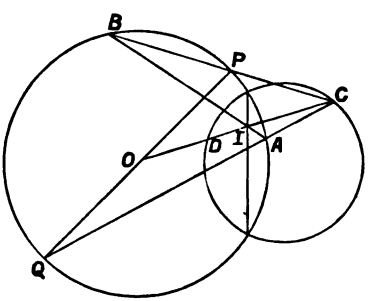
On lines drawn from the centre of the given polygon of $2n$ sides to the vertices as diameters, draw circles, and join their intersections so as to inscribe in each a similar polygon. The given polygon is thus divided into *rhombs*, arranged in $n-1$ concentric *zones*; or, omitting $n-2$ sides of each small polygon, into $2n$ symmetrical crescent-like *segments*, which can be combined in two ways. [The outline of a segment is that of a similar polygon *folded* along one of the diagonals which divide it least unequally.]

The extreme zones, or any pair equally distant from them respectively, have similar rhombs, and may be called *corresponding* zones. I suppose all figures to be so drawn that the rhombs of any zone, and hence the segments, are indistinguishable. The process is then definite, and its effect is (1) to interchange the rhombs of corresponding zones, (2) to turn every rhomb through a right angle, in the same sense, relatively to the radius drawn to its centre from the centre of the polygon. If the figure preserve throughout the *central* symmetry of the rhombs, both ways of cutting give the same result. If not, it is one of four figures so related that one mode of division turns each into the next in cyclical order, the other mode reversing the order. The condition that a figure may turn into its perversion is that it shall preserve (for the

mode of division chosen) the axial symmetry of the segments. To save space, in the subjoined examples, halves of mutually convertible figures are combined in the form of a single polygon. [We regret that we can afford space for only *one* of the very interesting illustrative diagrams sent by Mr. Hervey.]

10402. (E. M. Langley, M.A.) — If a diameter of one of two orthogonal circles cuts the common chord at I, and the other circle in C, and if AB is any chord of the first circle through I, show that CI is a symmedian of the triangle ABC.

Solution by Professor Genese, M.A.



Let O be the centre of the first circle, OC meet the second circle at D, and CB, CA meet the first circle at P, Q; then

$$OD \cdot OC = OP^2 = OD^2,$$

$\therefore \angle OPD = OCP = OBD$, hence O, D, P, B are concyclic, and $\angle POD = PBD$. So $\angle QOD = CAD$. Now

$$IA \cdot IB = ID \cdot IC;$$

hence A, B, C, D are concyclic, and angles CAD, CBD supplementary; $\therefore$ $\angle$'s POD, QOD are so, and P, O, Q are in a line. But PQ, AB are anti-parallel in angle ACB. Thus CI, a median to PQ, is a symmedian to AB.

10429. (A. E. Jolliffe.)—A parabola touches the sides of a triangle at A′, B′, C′; prove that AA′, BB′, CC′ meet on the minimum ellipse circumscribing the triangle.

Solution by Hon. Bertrand Russell; Prof. Anderson; *and others.*

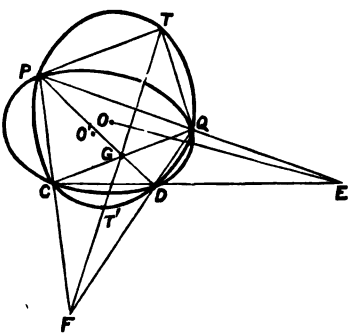
If the parabola be $(lx)^{\frac{1}{2}} + (my)^{\frac{1}{2}} + (nz)^{\frac{1}{2}} = 0$,

the intersection of AA′, BB′, CC′ is (l^{-1}, m^{-1}, n^{-1}). But $l+m+n=0$; hence the intersection lies on the conic $x^{-1}+y^{-1}+z^{-1}=0$, whose centre is $x=y=z$, *i.e.*, the centroid of the triangle. Hence the intersection lies on the minimum circumscribing ellipse.

[If we draw an equilateral triangle whose sides touch a given parabola, the lines that join any angular point with the point of contact of the opposite side will each pass through the focus, which is a point on the circumcircle of the triangle. Projecting orthogonally, &c.]

10432. (R. Knowles.)—Tangents TP, TQ are drawn to meet a conic in P and Q; a second conic, centre O, through TPQ, cuts the first again in C, D; T′ is the pole of CD with respect to the first conic; the chords PQ, CD meet in E; prove that EO bisects TT′.

Solution by R. H. Whapham, B.A.; D. T. Griffiths; *and others.*

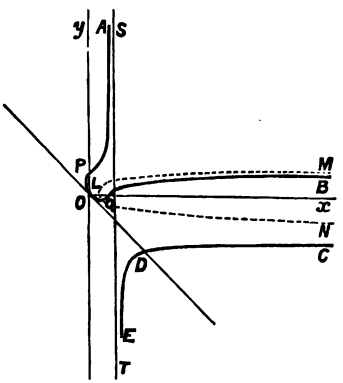


Let O′ be the centre of the first conic, and call the conics by their centres; then, we see that T′ is on the conic through T, P, Q, C, D. Now T is the pole of PQ, and T′ of CD; therefore E is the pole of TT′ with respect to conic O′. But, since PQDC is a quadrilateral inscribed in the conic O′, therefore E is the pole of GF with respect to conic O′, therefore TT′ is the same line as GF. But, since PQDC is also inscribed in the conic O, therefore E is the pole of GF with respect to the conic O, therefore E is the pole of TT′ with respect to the conic O, therefore EO bisects TT′.

10433. (H. J. Read, M.A.)—Trace the curve

$$(x+y)(x-y)^2-4\,(x^2+y^2)+5y=0.$$

Solution by Prof. Anderson, M.A.; G. G. Storr, M.A.; *and others.*



Turn the axes in the direction xy through an angle of 45°, and the equation of the curve becomes

$$xy^2-2^{\frac{1}{2}}(x^2+y^2)+\tfrac{5}{4}(x+y)=0.$$

The curve has two branches APOQB and CDE. The tangent at the origin is the line $x+y=0$, and the line ST, whose equation is $x=2^{\frac{1}{2}}$, is an asymptote. The parabola MLN, whose equation is

$$y^2=2^{\frac{3}{2}}(x-\tfrac{5}{16}2^{\frac{1}{2}}),$$

is a curvilinear diameter, bisecting all chords parallel to the axis of x, and consequently meets both branches at points where the tangent is parallel to this axis.

2508. (Professor CROFTON, F.R.S.)—(1) A point being denoted by (ρ, σ, τ), its tri-polar coordinates, or distances from three poles R, S, T, show that all circles represented by $A\rho^2 + B\sigma^2 + C\tau^2 = 0$ are orthogonal to the circle RST. (2) If O be the centre of the circle $A\rho^2 + B\sigma^2 + C\tau^2 = D$, show that A, B, C are proportional to the triangles SOT, TOR, ROS.

Solution by W. J. CURRAN SHARP, M.A.

This question may be generalised into—(1) A point in space of n dimensions being denoted by $(\rho, \sigma, \tau, \upsilon, \ldots)$, its $n+1$ polar coordinates, or distances from $n+1$ poles R, S, T, U ..., show that all spherics, represented by $A\rho^2 + B\sigma^2 + C\tau^2 + D\upsilon^2 + \ldots = 0$, are orthogonal to the circumspheric of the simplicissimum RSTU .... (2) If K be the centre of the spheric $A\rho^2 + B\sigma^2 + C\tau^2 + D\upsilon^2 + \ldots = M$, show that A, B, C, D ... are proportional to the contents of the simplicissima KSTU ..., KRTU ..., KRSU ..., KRST ..., &c. With notation, &c., of the solution of Quests. 9093 and 9170 [Vol. XLVIII., pp. 178–82],

$$\nabla^2\rho^2 = -S + \nabla\frac{dS}{d\lambda}, \quad \nabla^2\sigma^2 = -S + \nabla\frac{dS}{d\mu}, \ldots .$$

Therefore $$A\rho^2 + B\sigma^2 + C\tau^2 + D\upsilon^2 + \ldots = 0$$

is equivalent to

$$S - \frac{\nabla}{A+B+C+\ldots}\left\{A\frac{dS}{d\lambda} + B\frac{dS}{d\mu} + C\frac{dS}{d\nu} + \ldots\right\} = 0 \quad \ldots\ldots\ldots\ldots(1),$$

or $$S - \left(\lambda''\frac{dS}{d\lambda} + \mu''\frac{dS}{d\mu} + \nu''\frac{dS}{d\nu} + \ldots\right) = 0$$

if $$\lambda'' = A\nabla/(A+B+C\ldots), \quad \mu'' = B\nabla/(A+B+C+\ldots),$$

i.e., $$S - \left(\lambda\frac{dS''}{d\lambda''} + \mu\frac{dS''}{d\mu''} + \nu\frac{dS''}{d\nu''} + \ldots\right)\frac{\lambda+\mu+\nu\ldots}{\nabla} = 0,$$

which, compared with the equation

$$S - \left(\lambda\frac{dS'}{d\lambda'} + \mu\frac{dS'}{d\mu'} + \nu\frac{dS'}{d\nu'} + \ldots\right)\frac{\lambda+\mu+\nu\ldots}{\nabla} + \frac{\nabla^2r^2 + S'}{\nabla^2}(\lambda+\mu+\nu\ldots)^2 = 0 \quad \ldots\ldots\ldots\ldots(2),$$

to the spheric centre k $(\lambda', \mu', \nu'\ldots)$, and radius r, leads to

$$\frac{dS''}{d\lambda''} = \frac{dS'}{d\lambda'} - \frac{\nabla^2r^2 + S'}{\nabla}, \quad \frac{dS''}{d\mu''} = \frac{dS'}{d\mu'} - \frac{\nabla^2r^2 + S'}{\nabla}, \text{ &c.},$$

or $$-\nabla + \lambda'' + \mu'' + \nu'' + \ldots = 0,$$

$$-\frac{dS'}{d\lambda'} + \frac{\nabla^2r^2 + S'}{\nabla} + (1\,.\,2)^2\mu'' + (1\,.\,3)^2\nu'' + \ldots = 0,$$

$$-\frac{dS'}{d\mu'} + \frac{\nabla^2r^2 + S'}{\nabla} + (2\,.\,1)^2\lambda'' + (2\,.\,3)^2\nu'' + \ldots = 0,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

therefore $$\begin{vmatrix} V, & 1, & 1, & 1, & \dots \\ \frac{dS'}{d\lambda'} - \frac{V^2r^2+S'}{V}, & 0, & (1.2)^2, & (1.3)^2 \dots \\ \frac{dS'}{d\mu'} - \frac{V^2r^2+S'}{V}, & (2.1)^2, & 0, & (2.3)^2 \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 0,$$

and therefore $$\begin{vmatrix} 0, & 1, & 1, & 1, & \dots \\ \frac{V^2r^2+S'}{V}, & 0, & (1.2)^2, & (1.3)^2 & \dots \\ \frac{V^2r^2+S'}{V}, & (2.1)^2, & 0, & (2.3)^2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 0;$$

therefore $(V^2r^2+S')(-2)^n(n!)^2V = 0$, and $S' = -V^2r^2$;

but $S' = V^2(R^2 - Ok^2)$, where O is the circumcentre and R the circumradius, therefore $R^2 + r^2 = Ok^2$ which proves (1). (2) follows at once by comparing the sinister of (1) = M with (2), for

$$A : B : C : \dots :: \lambda' : \mu' : \nu' : \dots .$$

10410 & 10444. (Professor Curtis, M.A.) — (10410). Prove that there holds among any four points on an equilateral hyperbola the relation (analogous to Ptolemy's theorem) that if the points be 1, 2, 3, 4,

$$(12.34)^{\frac{1}{2}} \pm (23.14)^{\frac{1}{2}} \pm (31.24)^{\frac{1}{2}} = 0,$$

where 12 = the area of a right-angled triangle whose base is the line joining 1 and 2, and the sides of which are parallel to the asymptotes of the hyperbola.

(10444). Prove that the equation of an equilateral hyperbola touching three equilateral hyperbolas (the four curves being similarly placed) is

$$(23.S_1)^{\frac{1}{2}} \pm (31.S_2)^{\frac{1}{2}} \pm (12.S_3)^{\frac{1}{2}} = 0,$$

where 23 is the area of a triangle of which the base is a common tangent to S_2 and S_3, and the sides are parallel to the asymptotes.

Solution by W. J. Johnston, M.A.; Prof. Anderson, M.A.; *and others.*

If we substitute $x = \xi + \eta$, $y = (\xi - \eta)(-1)^{\frac{1}{2}}$, then $x^2 + y^2 = 4\xi\eta$, and all circles are transformed into similarly placed rectangular hyperbolas. If two curves touch, the corresponding curves touch at the corresponding point. Also, square of distance between two points $(x_1 - x_2)^2 + (y_1 - y_2)^2$ becomes $(\xi_1 + \eta_1 - \xi_2 - \eta_2)^2 - (\xi_1 - \eta_1 - \xi_2 + \eta_2)^2 = 4(\xi_1 - \xi_2)(\eta_1 - \eta_2)$ = 8 area of triangle whose sides are the join of the points and parallels through them to the asymptotes. From these considerations we can write down at once the above transformations of Ptolemy's theorem, and of the equation of the circle-tangent to three circles.

3657. (Professor MINCHIN, M.A.)—Show that the force by which a particle describes the inverse (with respect to centre of force) of a given curve is $\mu . r^5 (r^2 - 2p\rho) / p^3\rho$, where r is the radius-vector to any point on the given curve, ρ and p the radius to curvature and perpendicular to a tangent at the point.

Solution by Professor ANDERSON, M.A.

The force acting from the centre is $-\frac{h^2}{p'^3}\frac{dp'}{dr'}$, where p' and r' refer to the inverse curve; also $p' = r'p/r$, and $rr' = c^2$; hence, substituting for p' and r' in terms of p and r, we get

$$-\frac{h^2}{p'^3}\frac{dp'}{dr'} = -\frac{h^2 r^6}{c^6}\frac{1}{p^3}\left\{\frac{p}{r} - r\frac{d}{dr}\left(\frac{p}{r}\right)\right\} = -\frac{h^2 r^5}{c^6 p^3}(2p - r^2/\rho).$$

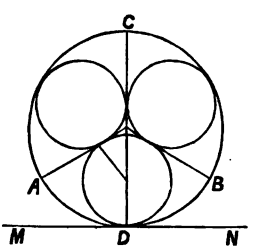
Hence the force is that stated in the question.

3302. (W. SIVERLY.)—Required the axes of the three greatest equal ellipses that can be drawn in an ellipse whose axes are $2a$ and $2b$.

Solution by Rev. J. L. KITCHIN, M.A.

Draw a circle ABC with diameter $2a$; take A, B, C the angular points of an equilateral triangle inscribed in the circle, AB being horizontal; inscribe in the three sectors the equal circles as in figure; and project the whole figure on a plane passing through MN, the tangent at D, at an angle whose cosine $= b/a$. The circle becomes an ellipse whose diameters are $2a$, $2b$, and the three equal inscribed circles, equal ellipses whose diameters are

$$2a(2\sqrt{3}-3), \quad 2b(2\sqrt{3}-3).$$



10329. (Prof. ORCHARD, M.A., B.Sc.)—Show that the roots of the equation

$$x^8 + (x^2 - x)^4 + (x^2 - 2x)^4 + (x^2 - 3x + 2)^4 + 9(x-1)^4 + 7(x-2)^4 + 16x^4 + 63 = 0$$

are $1 \pm (-3 \pm 7^{\frac{1}{2}})^{\frac{1}{2}}$, $\frac{1}{2}\left\{1 \pm \left[-3 \pm (-3)^{\frac{1}{2}}\right]^{\frac{1}{2}}\right\}$.

Solution by W. J. GREENSTREET, M.A.; Rev. J. L. KITCHIN; *and others.*

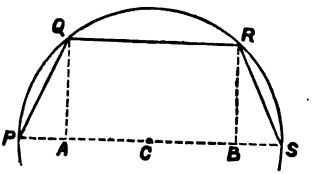
The equation resolves into the product of the two equations,

$$x^4 + 9 + (x-2)^4 = 0, \quad x^4 + 7 + (x-1)^4 = 0,$$

whose respective roots are the values given in the equation.

10217. (M. Leroux.)—Dans un trapèze isoscèle, on connaît une base et les côtés égaux. Quelle doit être l'autre base pour que l'aire du trapèze soit maximum?

Solution by Professor Sarkar; E. M. Langley, M.A.; *and others.*



Let PQ, RS be the two given equal sides, QR the other given side; then the greatest *quadrilateral* with PQ, QR, RS for sides, and the other side at our choice, is such that the fourth side PS is the diameter of a circle through P, Q, R, S. Since PQ = RS, this is a trapezoid, and therefore, since it is the greatest of all quadrilaterals, it must necessarily be the greatest of such of them as are trapezoids. To find PS we have only to bisect a straight line AB equal to QR in C, and produce it to points P and S, such that $CP^2 = CS^2 = CB^2 + \frac{1}{2}RS^2$. Then $PS \cdot SB = 2\,CS \cdot SB = 2CP^2 - 2CB^2 = RS^2$; hence PQS, PRS are right angles.

3799. (J. B. Saunders.)—Find the equation of the curve described by the extremities of a horizontal diameter of the air-bubble of Quest. 3707, supposing its centre to move in a vertical line.

Solution by D. Biddle.

From the solution of Quest. 3707 (Vol. liii., p. 27), it is easy to see that, in the case of fresh water,

$$y = \frac{1}{24}\left[\beta \Big/ \left\{\beta + \frac{a^2}{\cdot 000048}\left[\log 2r\,(a^2 - \cdot 000048\beta) - \log\left\{2r\,(a^2 - \cdot 000048\beta) - \cdot 000096rx + \cdot 000048x^2\right\}\right]\right\}\right]^{\frac{1}{2}}$$

where all the measurements are in feet, and where x is at zero on the surface of the water. In the case of sea-water,

$$y' = \frac{1}{24}\left[\beta' \Big/ \left\{\beta' + \frac{a'^2}{\cdot 000044}\left[\log 2r\,(a'^2 - \cdot 000044\beta') - \log\left\{2r\,(a'^2 - 000044\beta') - 000088rx + \cdot 000044x^2\right\}\right]\right\}\right]^{\frac{1}{2}}.$$

10312. (Professor Décamps.)—Par un point P pris sur le prolongement du diamètre BA d'un cercle C, on mène une sécante quelconque

PMM′; on joint le centre C aux milieux N, N′ des arcs AM, AM′; les droites CN, CN′ rencontrent en D et D′ la perpendiculaire menée au diamètre AB par le point P. Prouver que le produit PD, PD′ est constant, quelle que soit la position de la sécante.

Solution by C. Morgan, M.A., R.N.; R. Knowles, B.A.; *and others.*

$\angle NCA = \frac{1}{2}MCA = MBA = \alpha$,

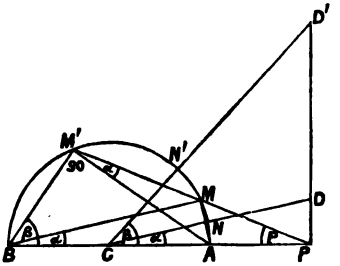
$\angle N'CA = \frac{1}{2}MCA = M'BA = \beta$;

$PD \cdot PD' = CP^2 \cdot \tan\alpha \cdot \tan\beta$;

also $\tan\alpha \cdot \tan\beta = \dfrac{AM}{BM} \cdot \dfrac{AM'}{BM'}$;

$AM = \dfrac{\sin P}{\sin\beta} \cdot AP$, $AM' = \dfrac{\sin P}{\sin\alpha} \cdot AP$,

$BM = \dfrac{\sin P}{\cos\beta} \cdot BP$, $BM' = \dfrac{\sin P}{\cos\alpha} \cdot BP$;



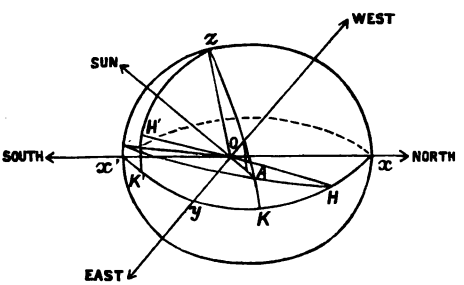
therefore $\tan\alpha \tan\beta = \dfrac{AP}{BP}$, therefore $PD \cdot PD' = CP^2 \cdot \dfrac{AP}{BP}$.

3729. (Professor Hudson, M.A.)—At noon, in latitude λ, the shadows upon a hill of the edges of a rectangular horizontal cloud are observed to be equally inclined at an angle γ to the horizon: regarding the hill as a plane inclined at an angle i to the horizon, with a north-westerly aspect, whose intersection with the horizon has an azimuth β east of north, find at what time of the year the observation took place. If $\lambda = 53° 28'$, $\beta = 30° = i$, and it was at midsummer, prove that

$$\gamma = \tan^{-1}(\tfrac{1}{4}\sqrt{3}).$$

Solution by the Proposer; Professor Anderson, M.A.; *and others.*

Take rectangular axes Ox, Oy, Oz pointing to the north, east, and zenith, respectively, and refer straight lines radiating from O to a sphere



with centre O; let HOH′ be the intersection of the plane of hill with the horizon, then $HOx = \beta$; and let OA, OA′ be the shadows, and let plane through 2A, 2A′ mark the horizontal plane in KK′, then AOK then $AHK = i$, and right-angled spherical triangle AHK, let $HK = \alpha$, $= A'OK' = \gamma$. In the $\sin \alpha = \tan \gamma \cot i$. Also, $HK' = \pi - \alpha$.

The direction-cosines of OA, OA′, and the sun's rays, are

$$\{\cos\gamma\cos(\alpha+\beta),\ \cos\gamma\sin(\alpha+\beta),\ \sin\gamma\},$$
$$\{-\cos\gamma\cos(\alpha-\beta),\ \cos\gamma\sin(\alpha-\beta),\ \sin\gamma\},\quad \{-\cos\theta,\ 0,\ \sin\theta\},$$

where θ is the sun's meridian altitude. The equation of the plane through OA, OA′ and the sun are

$$\sin\theta\cos\gamma\sin(\alpha+\beta)\,x - [\sin\theta\cos\gamma\cos(\alpha+\beta) + \cos\theta\sin\gamma]y + \cos\theta\cos\gamma\sin(\alpha+\beta)\,z = 0,$$

$$\sin\theta\cos\gamma\sin(\alpha-\beta)\,x + [\sin\theta\cos\gamma\cos(\alpha-\beta) - \cos\theta\sin\gamma]\,y + \cos\theta\cos\gamma\sin(\alpha-\beta)\,z = 0;$$

and their traces on the plane $z = 0$ are at right angles; hence we obtain

$$\tan^2\theta\cos^2\alpha + 2\tan\theta\tan\gamma\sin\alpha\sin\beta - \tan^2\gamma = 0,$$

an equation which determines θ, and thence the time of the year.

If $\beta = 30$, $i = 30$, $\theta = 60$, $\sin\alpha = \sqrt{3}\tan\gamma$, $\therefore \tan\gamma = \frac{1}{3}\sqrt{3}$.

10346. (Professor Neuberg.)—Etant données deux ellipses concentriques et homothétiques, on mène en un point quelconque M de l'ellipse intérieure la normale qui rencontre les axes en P, Q; la tangente en M rencontre l'ellipse extérieure en R, S. Démontrer que les angles RPS, RQS sont constants.

Solution by Rev. T. Gallibers, M.A.; G. G. Storr, M.A.; *and others.*

Let the equations of the inner and outer ellipses be $b^2x^2 + a^2y^2 = a^2b^2$, $b^2x^2 + a^2y^2 = \mu^2a^2b^2$, and let ϕ be the excentric angle of M, and $a^2\sin^2\phi + b^2\cos^2\phi = d^2$; then the equation of the tangent at M is

$$(x - a\cos\phi)/(a\sin\phi) = (y - b\sin\phi)/(-b\cos\phi) = r/d;$$

hence the equation to determine MR, MS is

$$a^2b^2(d\cos\phi + r\sin\phi)^2\,\theta\,(\cos\phi + 2\sin\phi)^2 + a^2b^2(d\sin\phi - r\cos\phi)^2 = \mu^2a^2b^2d^2,$$

or $$r = \pm(\mu^2-1)^{\frac{1}{2}}d;\quad \text{that is,}\quad MR = MS.$$

Again, $PM = b\,.\,d/a$, also $QM = a\,.\,d/b$;

hence $$\tan\tfrac{1}{2}RPS = MR/PM = a(\mu^2-1)^{\frac{1}{2}}/b,$$
$$\tan\tfrac{1}{2}RQS = MR/QM = b(\mu^2-1)^{\frac{1}{2}}/a.$$

Thus both RPS and RQS are constants. It is clear, also, that the triangles RPS, RQS are isosceles, and are bisected symmetrically by the normal to the inner ellipse at M.

7715. (W. J. C. Sharp, M.A.)—Two bags contain, the one n white balls and the other n black; and each set is numbered $1 . 2 \ldots n$. The balls are drawn in pairs, one from each bag; find the chance, at any stage, that no pair has been drawn consisting of balls bearing the same number; and show that, as the number drawn increases, this approaches to e^{-1}.

Solution by Professor Schoute.

We may suppose that the balls drawn from the first bag appear as the numbers 1, 2, ... n, in natural order. For, if this is not the case, a simple permutation of numbers will make it so. Therefore the problem can be thrown into the following more convenient form:—

"A bag contains n balls B_1, B_2, ... B_n, numbered 1, 2, ... n. Find the chance that in k drawings no ball has been drawn bearing a number equal to the order of the drawing, and show that this chance approaches to e^{-1} as the numbers k and n increase."

The number of all possible cases is $1 . 2 \ldots n$, say P_n.

The number of cases the first ball drawn is B_1 is P_{n-1}.

The number of cases the second ball drawn is B_2 (without the first having been B_1) is $P_{n-1} - P_{n-2}$.

The number of cases the third ball drawn is B_3 (without the first having been B_1 or the second B_2) is $P_{n-1} - 2P_{n-2} + P_{n-3}$, &c. Thus we find

$$P_{n-1}, \quad P_{n-1} - P_{n-2}, \quad P_{n-1} - 2P_{n-2} + P_{n-3},$$
$$P_{n-1} - 3P_{n-2} + 3P_{n-3} - P_{n-4}, \quad \ldots\ldots\ldots$$

and, with the notation of binomial coefficients, at last

$$P_{n-1} - \binom{n-1}{1} P_{n-2} + \binom{n-1}{2} P_{n-3} - \binom{n-1}{3} P_{n-4} + \ldots$$

The sum of the first k lines is

$$\binom{k}{1} P_{n-1} - \binom{k}{2} P_{n-2} + \binom{k}{3} P_{n-3} - \binom{k}{4} P_{n-4} + \ldots,$$

or
$$P_n \left\{ \frac{k}{n} - \frac{1}{1.2} \frac{k(k-1)}{n(n-1)} + \frac{1}{1.2.3} \frac{k(k-1)(k-2)}{n(n-1)(n-2)} - \ldots \right\}.$$

So the chance $W_{k,n}$ in question is given by the equation

$$W_{k,n} = 1 - \frac{1}{1} \frac{k}{n} + \frac{1}{1.2} \frac{k(k-1)}{n(n-1)} - \frac{1}{1.2.3} \frac{k(k-1)(k-2)}{n(n-1)(n-2)} + \ldots,$$

which proves that $W_{k,n}$ approaches to $1/e$ when k and n increase.

[For an analogous problem, Professor Schoute refers to *Wiskundige Opgaven*, tome iv., page 96, problem 34.]

10244. (F. R. J. Hervey.)—Prove that the inverses of a point P with respect to four mutually orthogonal circles, invert from P and from three other points into mutually orthocentric points.

Solution by Professors MUKHOPÂDHYÂY, BEYENS, *and others.*

It is known that a circle and a pair of inverse points with respect to that circle invert from any arbitrary point into a circle and a pair of inverse points with respect to this latter circle. (CASEY'S *Sequel*, M. Ex. 60.)

Let P_1, P_2, P_3, P_4 be the inverses of the point P with respect to the four mutually orthogonal circles A, B, C, D. Invert the system from P. The four mutually orthogonal circles invert into four other mutually orthogonal circles A′, B′, C′, D′. The point P itself is inverted into a point at infinite distance. Hence, from the assumption above-mentioned, the points P_1, P_2, P_3, P_4 must invert into the centres of A′, B′, C′, D′, that is to say, into four mutually orthocentric points. Each of the four points P_1, P_2, P_3, P_4, when inverted with respect to the circles A, B, C, D, gives the same four points P, Q, R, S, and therefore each of the three other points Q, R, S, when inverted with respect to A, B, C, D, gives the same four points P_1, P_2, P_3, P_4. This will be clear if we bear in mind that any point is brought back to its old position by successive inversions in any order with respect to four mutually orthogonal circles. (CASEY'S *Sequel to Euclid*, Sec. iv., Prop. 16.)

[The eight points P, Q, R, S, P_1, P_2, P_3, P_4 have this property, that twelve circles can be drawn through them, four points on each circle, and six circles (in three orthogonal pairs) through each point. The relations may be conveniently studied by considering the *real* circle corresponding to the *imaginary* given circle as a great circle of a sphere, upon which the eight points will then project stereographically into the vertices of a right solid; either set, P, Q, R, S or P_1, P_2, P_3, P_4, becoming ends of non-parallel diagonals of opposite faces.]

9619. (Professor SIRCOM, M.A.)—Find the locus of the vertices of a triangle of given species, self-conjugate with respect to a given ellipse. Discuss the various cases, and in particular examine the case in which the triangle is equilateral.

Solution by W. J. GREENSTREET, M.A.; Prof. NASH, M.A.; *and others.*

If the conic be $(x^2/a^2) + (y^2/b^2) = 1$,

and $x\cos\alpha + y\sin\alpha = 0, \quad x\cos\beta + y\sin\beta = 0$

be lines through origin parallel to the sides of the triangle meeting in x, y; we have, since these sides and the tangents from x, y to the curve form an harmonic pencil,

$$\cos\alpha\cos\beta\,(x^2-a^2) + \sin\alpha\sin\beta\,(y^2-b^2) - xy\sin(\alpha+\beta) = 0;$$

i.e., $$(x^2+y^2-a^2-b^2)\cos(\alpha-\beta) - (x^2-y^2-c^2)\cos(\alpha+\beta) - 2xy\sin(\alpha+\beta) = 0 \quad \ldots\ldots\ldots\ldots(1).$$

And $\alpha+\beta = 2\omega$, $\cos\omega = \dfrac{x^2}{a}\Big/\left(\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}\right)^{\frac{1}{2}}$, $\sin\omega = \dfrac{y}{b^2}\Big/\left(\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}\right)^{\frac{1}{2}}$;

therefore, substituting these values, we have

$$(x^2+y^2-a^2-b^2)\cos(\alpha-\beta) - (x^2-y^2-c^2)\cos 2\omega - 2xy\sin 2\omega = 0.$$

In an equilateral triangle $\alpha-\beta = 60°$; therefore, substituting for $\cos 2\omega$, $\sin 2\omega$, $\cos(\alpha-\beta)$, we get

$$\tfrac{1}{2}(x^2+y^2-a^2-b^2)\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right) = (x^2-y^2-c^2)\left(\frac{x^2}{a^4}-\frac{y^2}{b^4}\right)+\frac{4x^2y^2}{a^2b^2},$$

or
$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2-\frac{3c^4x^2y^2}{a^4b^4}+(3b^2-a^2)\frac{x^2}{a^4}+(3a^2-b^2)\frac{y^2}{b^4} = 0.$$

10279. (Professor Orchard, M.A., B.Sc.)—Solve the simultaneous differential equations $\frac{d^2x}{dt^2}-2y-x = 3$, $\frac{d^2y}{dt^2}-5y-4x = 6$.....(1, 2).

Solution by W. J. Greenstreet, M.A.; Prof. Cavallin, M.A.; *and others.*

$$\left[(1)\times\left\{\left(\frac{d}{dt}\right)^2-5\right\}+(2)\times 2\right]$$

gives
$$\left\{\left(\frac{d}{dt}\right)^2-5\right\}\left\{\left(\frac{d}{dt}\right)^2-1\right\}x-8x = -3,$$

or
$$\left\{\left(\frac{d}{dt}\right)^2+a^2\right\}\left\{\left(\frac{d}{dt}\right)^2+b^2\right\}x = -3,$$

where a^2, b^2 are the roots of $z^2+6z-3 = 0$; whence

$$x = 1+C_1\cos(ax+\alpha)+C_2\cos(bx+\beta),$$
$$y = -2-\tfrac{1}{2}(a^2+1)C_1\cos(ax+\alpha)-\tfrac{1}{2}(b^2+1)C_2\cos(bx+\beta).$$

10337. (H. W. Segar.)—A fixed point P is taken on the auxiliary circle of a rectangular hyperbola, and Q, R are the ends of any diameter of the conic. O is the centre of the circle PQR; O has a certain locus. Find, whatever be the position of P, the equations to the tangents at the centre of the conic to these loci.

Solution by Rev. T. R. Terry, M.A.; Rev. T. Galliers, M.A.; *and others.*

Let P, Q be $(a\cos\alpha, a\sin\alpha)$, $(a\cos\theta\sec^{\frac{1}{2}}2\theta, a\sin\theta\operatorname{se}c^{\frac{1}{2}}2\theta)$; then, if the equation to the circle PQR is $X^2+Y^2+AX+BY+C = 0$, the centre O is given by $2x+A = 0$, $2y+B = 0$; and A, B, C are given by $a^2+Aa\cos\alpha+Ba\sin\alpha+C = 0$, $a^2\sec 2\theta \pm (Aa\cos\theta+Ba\sin\theta)\sec^{\frac{1}{2}}2\theta+C = 0$. From this we have $\tan\theta = A/B = x/y$, also

$$-a^2(1-\cos 2\theta)\sec 2\theta+Aa\cos\alpha+Ba\sin\alpha = 0;$$

hence the locus of O is $ax^2+(x\cos\alpha+y\sin\alpha)(y^2-x^2) = 0$, the tangents to which at the origin are $x^2 = 0$.

9239. (Professor NEUBERG.)—Trouver l'enveloppe d'un plan tel que les symétriques de ce plan, par rapport aux faces d'un tétraèdre donné, ABCD, concourent en un même point.

Solution by Professor SCHOUTE.

With reference to a rectangular system the four given planes P_1, P_2, P_3, P_4, and any other plane P, can be represented by ($i = 1, 2, 3, 4$)

$$P_i \equiv u_i x + v_i y + w_i z + t_i = 0, \quad P \equiv ux + vy + wz + t = 0.$$

Now $P - k_i P_i = 0$ represents a plane through the intersection of P and P_i. The distance of the point (x, y, z) of P_i to this plane is

$$P / \left\{(u - k_i u_i)^2 + (v - k_i v_i)^2 + (w - k_i w_i)^2\right\}^{\frac{1}{2}},$$

which will be equal to the distance of the same point to P under the condition $(u - k_i u_i)^2 + (v - k_i v_i)^2 + (w - k_i w_i)^2 = u^2 + v^2 + w^2$.

So we find $2(uu_i + vv_i + ww^i) = k_i (u_i^2 + v_i^2 + w^2)$,

and, if $u_i^2 + v_i^2 + w_i^2 = F_i, \quad uu_i + vv_i + ww_i = G_i$,

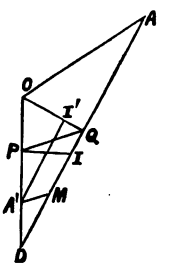
the condition is

$$\begin{vmatrix} uF_1 - 2u_1G_1, & vF_1 - 2v_1G_1, & wF_1 - 2w_1G_1, & tF_1 - 2t_1G_1 \\ uF_2 - 2u_2G_2, & vF_2 - 2v_2G_2, & wF_2 - 2w_2G_2, & tF_2 - 2t_2G_2 \\ uF_3 - 2u_3G_3, & vF_3 - 2v_3G_3, & wF_3 - 2w_3G_3, & tF_3 - 2t_3G_3 \\ uF_4 - 2u_4G_4, & vF_4 - 2v_4G_4, & wF_4 - 2w_4G_4, & tF_4 - 2t_4G_4 \end{vmatrix} = 0.$$

As u, v, w, t are the homogeneous tangential coordinates of P, this plane envelopes a surface of the fourth class. The equation of this surface can be transformed into $(u_1 v_2 w_3 t_4) + \Sigma \frac{u}{G_1} (F_1 v_2 w_3 t_4) = 0$,

the first member of which consists of five determinants.

10316. (Prof. W. S. MCCAY, M.A.)—Let P, Q be the feet of the perpendiculars from the incentre of a triangle on the perpendicular bisector of the base, and from the circumcentre on the bisector of the vertical angle. Prove that PQ passes through the incentre of the triangle formed by the mid-points of the sides of the given triangle.



Solution by Professor RAMASWAMI AIYAR; D. T. GRIFFITHS; *and others.*

Let ABC be the given triangle; A', B', C' the middle points of its sides; O its circumcentre, and I the incentre.

Through A draw A'I' parallel to AI, meeting PQ in I'; I' is the incentre of the triangle A'B'C'.

This is proved if we show that $A'I' = \frac{1}{2}AI$.

Let OA′ and AI meet at D; bisect DI at M. Then, since

$$DI^2 = (DB \text{ or } DC)^2 = \text{diameter} \times DA',$$

therefore $DM \cdot DI = DO \cdot DA'$;

but $DQ \cdot DI = DO \cdot DP$,

therefore $DM : DQ$ as $DA' : DP$;

and therefore PQ parallel to A′M. Hence

$$A'I' = MQ = DQ - DM = \tfrac{1}{2}DA - \tfrac{1}{2}DI = \tfrac{1}{2}AI.$$

[The PROPOSER remarks that this theorem results from the theory of "Three Similar Figures."

If the incentre I be taken for the extremity of the diameter Ok of the generalised Brocard-circle of the triangle ABC (circle of similitude), the ratios of similitude $u_1 : u_2 : u_3$ become 1 : 1 : 1, the feet of the perpendiculars from I on the perpendicular bisectors of the sides are the invariable points P_1, P_2, P_3; the feet of perpendiculars from O, the circumcentre, on the internal bisectors of the angles of ABC, are the double points S_1, S_2, S_3; P_1S_1, P_2S_2, P_3S_3 intersect in E the director-point; E is the double point of the triangle $P_1P_2P_3$ and the middle-point-side triangle of ABC and its distances from the sides of either of these triangles are as 1 : 1 : 1; hence the result in question. See *Transactions of the Royal Irish Academy*, Vol. xxix., p. 306, sec. 4.]

10372. (J. J. BARNIVILLE.)—If the normal at any point of a conic divide the abscissa in the ratio $1 : [1-(1-\&c.)^{\frac{1}{2}}]^{\frac{1}{2}}$, prove that the area of the curve is equal to that of a concentric circle through the foci.

Solution by W. J. GREENSTREET, M.A.; Professor SARKAR; *and others.*

By Question 10179,

$$CG/GN = 1 \Big/ \left(2 \sin \frac{\pi}{10}\right); \quad \therefore CG/CN = \frac{\sqrt{5}+1}{\sqrt{5}+3} = e^2.$$

Hence we easily obtain $e^2 = b/a$. Now area of concentric circle through the foci is $\pi a^2 e^2 = \pi ab$ = area of curve.

8942. (W. J. C. SHARP, M.A.)—The polar triangles of the spherical triangles, determined by the four solid angles of a tetrahedron, exactly occupy the surface of the sphere. Construct these, and hence prove the above.

Solution by Professor SCHOUTE.

Let O (see the diagram of Question 8943) be the incentre of the given tetrahedron, and Oa, Ob, Oc, Od perpendicular to the faces BCD, CDA, DAB, ABC. Then the solid angles O (bcd), O (cda), O (dab), O (abc) correspond to the polar triangles in question, and fill evidently the space round O. Hence, &c.

4054. (Rev. T. MITCHESON, B.A.)—In a spherical triangle, if t = the product of the tangents of the radii of the escribed circles, prove that

$$\tan r = (n^4 t^{-1} \sin^{-2} s)^{\frac{1}{3}}.$$

Solution by G. G. STORR, M.A.; Professor ZERR, M.A.; *and others.*

With the usual notation, we have $\tan r_1 = n/\sin(s-a)$, &c.;

hence $t = n^3 / \{\sin(s-a)\sin(s-b)\sin(s-c)\} = n \sin s$;

therefore $n^4 t^{-1} \sin^{-2} s = (n/\sin s)^3 = \tan^3 r$.

10411. (Professor WOLSTENHOLME, M.A., Sc.D.)—On a side BC of a given triangle ABC are described two equilateral triangles aBC, a'BC; a, A being on the same side of BC; prove that Aa is three times the difference, Aa' three times the sum of the semi-axes of the maximum ellipse inscribed in the triangle ABC, and that the axes of this ellipse are parallel to the bisectors of the angles between Aa, Aa'.

Solution by Professors ANDERSON, CHAKRAVARTI, *and others.*

Take for axes of coordinates the base BC of the triangle, and the perpendicular through its middle point, and let the length of the base be $2a$, and the coordinates of A, x', y'; then the equation of the ellipse is easily seen to be

$$y'^2x^2 - 2x'y'xy + (3a^2 + x'^2)y^2 - 2a^2yy' = 0.$$

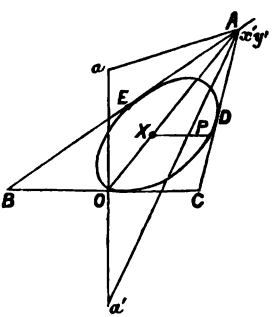
The axes are therefore parallel to the lines

$$x'y'(y^2 - x^2) - (y'^2 - x'^2 - 3a^2)xy = 0.$$

It follows that the axes are parallel to the bisectors of the angles between the lines Aa and Aa'; for their equations are

$$(xy' - yx')^2 - 3a^2(x - x')^2 = 0,$$

and their bisectors are likewise parallel to the lines $x'y'(y^2 - x^2) - (y'^2 - x'^2 - 3a^2)xy = 0$.



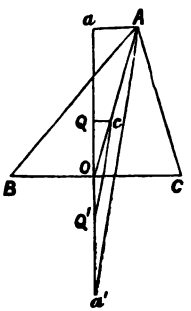
The centre of the ellipse is X, the centre of gravity of ABC. Draw XP parallel to the base BC. Then OX and XP are semi-conjugate diameters, and OX = $\frac{1}{3}$OA; also XP = $\frac{1}{3}$Oa, which is easily seen by transforming to the centre and making $y = 0$.

Now $(Aa')^2 - (Aa)^2 = 4a\text{O} \cdot \text{OA} \cos a\text{OA} = 36\,\text{OX} \cdot \text{XP} \sin \text{OXP} = 36\alpha\beta$, where α, β are the semi-axes major and minor. Also,

$$(Aa')^2 + (Aa)^2 = 2\left[(a\text{O})^2 + (\text{OA})^2\right] = 18\left[(\text{OX})^2 + (\text{XP})^2\right] = 18(\alpha^2 + \beta^2);$$

hence $Aa' = 3(\alpha + \beta)$, $Aa = 3(\alpha - \beta)$.

[The diameter of the maximum inscribed conic which is parallel to BC is two-thirds of Oa, or Oa', as is obvious by orthogonal projection of the circle inscribed in an equilateral triangle. Hence, if Q, Q′ be taken on Oa, Oa', so that $Oa = 3OQ$, $Oa' = 3OQ'$, OQ, OQ′ are lengths measured along the normal at O equal to the semi-diameter conjugate to GO, and by the ordinary construction for the lengths and direction of the axes of an ellipse when two conjugate diameters are given, GQ is equal to the difference and GQ′ to the sum of the semi-axes; also, the major axis bisects the angle OGQ, and the minor axis the angle OGQ′. But GQ is parallel to Aa and equal to one-third of Aa; GQ′ is parallel to Aa' and equal to one-third of Aa'; whence the theorems stated. These give, apparently, the simplest construction for the principal axes of a triangular lamina at its centroid.

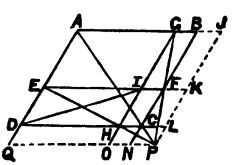


An equivalent construction is given in the third edition of Routh's *Rigid Dynamics*.]

10204. (J. C. St. Clair.)—Through each vertex of a triangle are drawn pairs of lines parallel to two fixed lines OX, OY; prove that (1) the three diagonals of the parallelograms having the sides of the triangle for their other diagonals meet in the same point P; and (2) if the sides of the constant angle XOY vary in direction, find the locus of P.

Solution by J. J. Barniville; Professor Dupain; *and others.*

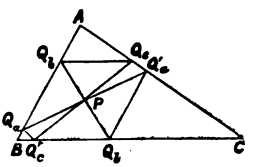
Let DIB be the triangle, and let EH, GF, AC be the other diagonals. Let EH, GF meet in P, and through P draw PQ, PJ parallel to the axes. Then the complement OF = FJ, therefore OFL = CJ (K and L being the intersections of EF, DC with PJ). Also, QH = HK; therefore QC = OFL. Hence QC = CJ, and ACP is the diagonal of the parallelogram AP.



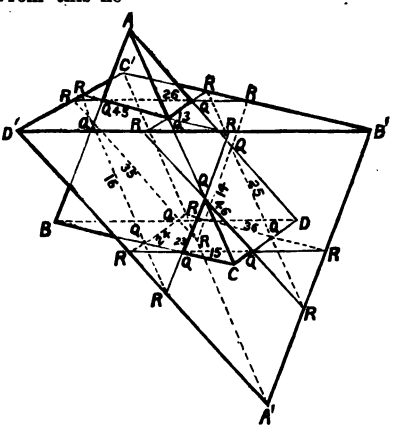
10412. (Professor Neuberg.)—Etant donnés un tétraèdre ABCD et un point P, on mène par P un plan coupant l'angle solide A suivant un triangle T_1, ayant pour centre de gravité P. Soient, de même, T_2, T_3, T_4 des triangles ayant pour centre de gravité P et leurs sommets sur les arêtes des trièdres B, C, D du tétraèdre. Démontrer que les quatre triangles T_1, T_2, T_3, T_4 sont inscrits à une même surface du second ordre.

Solution by the EDITOR.

Of the several long and interesting solutions that have been to this question, we regret that, from want of space, we can only give the following synopsis:—



1. Prof. SCHOUTE begins with the solution of the analogous problem in plane geometry, supposing the segments Q_a, Q'_a, Q_b, Q'_b limited by the sides of the triangle ABC to have a common mid-point P, and he proves, in several different ways, that the six points Q lie on a conic. From this he goes on at great length to discuss the general problem, adding at the end the following further development:—



If the homologous faces of the two tetrahedra ABCD and A'B'C'D' meet in four complanar lines, the other twelve lines of intersection lie on a cubic surface. This is *an* extension of PASCAL's theorem on space. Now we learn that these twelve lines of the cubic surface form in two manners four triangles, the vertices of which lie on two quadrics. In the simple case of parallel homologous faces, these two sets of twelve points Q and R are indicated in the diagram.

When, in the usual manner, the 27 lines of a cubic surface are represented by the symbols

$$a_1, a_2, a_3, a_4, a_5, a_6; \quad b_1, b_2, b_3, b_4, b_5, b_6; \quad c_{12}, c_{13}, \ldots\ldots c_{56},$$

we obtain a set of lines equivalent to the twelve of the diagram by diminishing the fifteen lines c by three complanar lines; *e.g.*, c_{12}, c_{34}, c_{56}. This is indicated in the diagram. As a cubic surface contains 36 double-six, and every completing set of 15 lines admits 15 triples of complanar lines, a cubic surface admits 36 . 15 . 2, or 1080, quadrics that pass in the indicated manner through twelve points in two lines.

2. Mr. SHARP and the other Solvers mostly proceed somewhat as follows:—Defining a point by the tetrahedral (content) coordinates λ, μ, ν, π, so that $\lambda+\mu+\nu+\pi = V$, where V is the content of ABCD. If P be $(\bar{\lambda}, \bar{\mu}, \bar{\nu}, \bar{\pi})$, and $(\lambda_2, \mu_2, 0, 0)$, $(\lambda_3, 0, \nu_3, 0)$, $(\lambda_4, 0, 0, \pi_4)$ the points where T_1 meets AB, AC, AD, respectively, $\lambda_2+\lambda_3+\lambda_4 = 3\bar{\lambda}$, $\mu_2 = 3\bar{\mu}$, $\nu_3 = 3\bar{\nu}$, $\pi_4 = 3\bar{\pi}$, and the points are

are $\quad (V-3\bar{\mu}, 3\bar{\mu}, 0, 0), (V-3\bar{\nu}, 0, 3\bar{\nu}, 0), (V-3\bar{\pi}, 0, 0, 3\bar{\pi})$.

Similarly, the points of intersection of T_2, T_3, T_4 with the edges through

B, C, D, respectively, are

$$(3\bar{\lambda},\ \nabla-3\bar{\lambda},\ 0,\ 0),\ (0,\ \nabla-3\bar{\nu},\ 3\bar{\nu},\ 0),\ (0,\ \nabla-3\bar{\pi},\ 0,\ 3\bar{\pi}),$$

$$(3\bar{\lambda},\ 0,\ \nabla-3\bar{\lambda},\ 0),\ (0,\ 3\bar{\mu},\ \nabla-3\bar{\mu},\ 0),\ (0,\ 0,\ \nabla-3\bar{\pi},\ 3\bar{\pi}),$$

$$(3\bar{\lambda},\ 0,\ 0,\ \nabla-3\bar{\lambda}),\ (0,\ 3\bar{\mu},\ 0,\ \nabla-3\bar{\mu}),\ (0,\ 0,\ 3\bar{\nu},\ \nabla-3\bar{\nu}),$$

all of which lie on the quadric surface.

$$\left(\frac{\nabla}{3\bar{\lambda}}-1\right)\lambda^2+\left(\frac{\nabla}{3\bar{\mu}}-1\right)\mu^2+\left(\frac{\nabla}{3\bar{\nu}}-1\right)\nu^2+\left(\frac{\nabla}{3\bar{\pi}}-1\right)\pi^2$$
$$-\left(\frac{\nabla}{3\bar{\lambda}}+\frac{\nabla}{3\bar{\mu}}-2\right)\lambda\mu-\left(\frac{\nabla}{3\bar{\lambda}}+\frac{\nabla}{3\bar{\nu}}-2\right)\lambda\nu-\left(\frac{\nabla}{3\bar{\lambda}}+\frac{\nabla}{3\bar{\pi}}-2\right)\lambda\pi$$
$$-\left(\frac{\nabla}{3\bar{\mu}}+\frac{\nabla}{3\bar{\nu}}-2\right)\mu\nu-\left(\frac{\nabla}{3\bar{\nu}}+\frac{\nabla}{3\bar{\pi}}-2\right)\nu\pi-\left(\frac{\nabla}{3\bar{\pi}}+\frac{\nabla}{3\bar{\mu}}-2\right)\mu\pi=0.$$

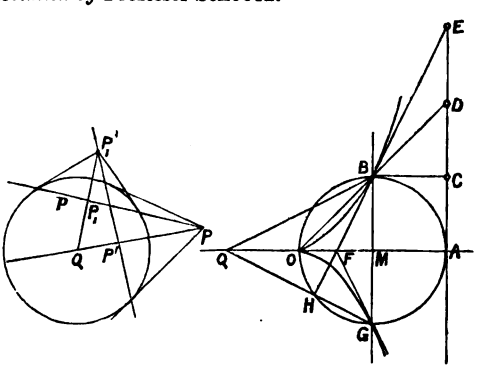
In a similar way it may be proved that "if, in space of n dimensions, linear loci be drawn through a given point to cut all the edges of the simplicissimum which meet in each vertex in points the centre of gravity of which is the given point, these points of intersection all lie upon a quadric locus." (See *Proc. Lond. Math. Soc.*, Vol. XVIII., pp. 325–59, and XIX., pp. 423–82.)

If $n = 2$, the property is, that if lines bisected in P meet the sides BC, CA, AB of a triangle in the points B_1, C_1; C_2, A_2; A_3, B_3, these six points lie on a conic.

9136. (N'IMPORTE, B.Sc.) — Find (1) a class of curves such that the pedal with regard to a given point is the inverse with regard to that point; and show (2) that the rectangular hyperbola is a particular case.

Solution by Professor SCHOUTE.

1. When P is a point of the curve, and Q the given point, the inverse point P′ of P with respect to the circle (Q) only can be a point of the pedal of the curve for Q if this curve is touched by the normal in P′ on PQ, that is, the polar of P with reference to Q. And if p is a tangent of the curve, the corresponding point P_1 of the pedal for Q only can be a point of the inverse



curve for Q if the pole P_1' of p with respect to (Q) is a point of the curve. Any curve that satisfies the conditions of the problem is therefore its own reciprocal polar with respect to Q. And, inversely, any curve that is its own reciprocal polar with respect to Q gives a solution of the problem.

2. The curve of order n and class m, with d nodes, k cusps, t double tangents, and i inflexional tangents, cannot be a solution unless $m = n$, $t = d$, $i = k$. So we find, according to Plücker's formulas, the cases

$$(a)\ m = n = 2,\quad t = d = 0,\quad i = k = 0;$$
$$(b)\ m = n = 3,\quad t = d = 0,\quad i = k = 1;$$
$$(c)\ m = n = 4,\quad t = d = 1,\quad i = k = 2;\ \&c.$$

3. In the case (a) the two tangents of the conic that contain Q correspond to the infinite points of the conic. Therefore Q is inside, outside, or on the conic, in the cases that this curve is an ellipse, a hyperbola, or a parabola, respectively. The only possible solutions are the circle Q itself, the rectangular hyperbola of which Q is the centre, and any diameter of Q the real axis. To these may be added the parabola of which Q is the vertex; but in this case the square of the radius of Q is to be taken affected with the negative sign, the parameter of the parabola being equal to that radius multiplied by $\sqrt{-1}$.

Of the other possible cases we only indicate that of the cissoid of Diocles. According to the ingenious construction of the tangent by reciprocal transversals due to De Longchamps, the tangent in B bisects OM in F (as CD = DE). Now the line GH through G normal to BF meets the axis OA in Q for which QO = OM = MA. And it is easily proved that the cissoid is its own reciprocal polar with reference to the circle Q, the radius of which is the geometric mean between QO and QA, or QH and QG, or QF and QM, &c.

2925. (J. J. Walker, F.R.S.)—Prove that the maximum value of the angle of intersection with an ellipse at D, of the circle passing through three points on the ellipse, the osculating circles at which meet in D, is $\sin^{-1}\frac{a^2-b^2}{2(a^2+b^2)}$; and that in this case the centre of the circle lies on the chord common to the ellipse and its osculating circle at D.

Solution by Professor Anderson, M.A.

Let δ denote the eccentric angle of the point D. The other three points will be $\frac{1}{3}(2\pi-\delta)$, $\frac{1}{3}(4\pi-\delta)$, $\frac{1}{3}(6\pi-\delta)$, and the circle through these points passes through D since $\delta+\frac{1}{3}(2\pi-\delta)+\frac{1}{3}(4\pi-\delta)+\frac{1}{3}(6\pi-\delta)=4\pi$. The equation of the chord joining δ and $\frac{1}{3}(6\pi-\delta)$ is

$$x/a\cos\tfrac{1}{3}\delta + y/b\sin\tfrac{1}{3}\delta - \cos\tfrac{2}{3}\delta = 0,$$

and the equation of chord joining $\frac{1}{3}(2\pi-\delta)$ and $\frac{1}{3}(4\pi-\delta)$ is

$$x/a\cos\tfrac{1}{3}\delta - y/b\sin\tfrac{1}{3}\delta + \tfrac{1}{2} = 0.$$

The equation of the circle is therefore

$$x^2/a^2+y^2/b^2-1+k\,(x/a\cos\tfrac{1}{2}\delta+y/b\sin\tfrac{1}{2}\delta-\cos\tfrac{3}{2}\delta)$$
$$(x/a\cos\tfrac{1}{2}\delta-y/b\sin\tfrac{1}{2}\delta+\tfrac{1}{2})=0,$$

where $k=(a^2-b^2)/(b^2\cos^2\tfrac{1}{2}\delta+a^2\sin^2\tfrac{1}{2}\delta)$; and this becomes

$$x^2+y^2-1+\tfrac{1}{2}(a^2-b^2)\left\{-x/a\cos\tfrac{1}{2}\delta(2\cos\tfrac{3}{2}\delta-1)\right.$$
$$\left.+y/b\sin\tfrac{1}{2}\delta\,(2\cos\tfrac{3}{2}\delta+1)-\cos\tfrac{3}{2}\delta\right\}=0.$$

The tangent of the angle which the tangent to this circle at the point δ makes with the axis of x is easily found to be $-b/a\cot\delta\,(3a^2+b^2)/(3b^2+a^2)$, and the tangent of the angle which the tangent to the ellipse at the same point makes with the axis of x is $-b/a\cot\delta$. Hence, if θ denote the angle between the tangents,

$$\tan\theta=-\frac{b}{a}\left\{\cot\delta\left(1-\frac{3a^2+b^2}{3b^2+a^2}\right)\right\}\Big/\left(1+\frac{b^2}{a^2}\cot^2\delta\,\frac{3a^2+b^2}{3b^2+a^2}\right)$$
$$=\frac{2ab\,(a^2-b^2)}{a^2\,(3b^2+a^2)\tan\delta+b^2\,(3a^2+b^2)\cot\delta}.$$

This is a maximum when

$$\tan\delta=\frac{b}{a}\left(\frac{3a^2+b^2}{3b^2+a^2}\right)^{\frac{1}{2}};\ \text{and then}\ \tan\theta\,\frac{a^2-b^2}{\left\{(3a^2+b^2)(3b^2+a^2)\right\}^{\frac{1}{2}}},$$

whence $\sin\theta=(a^2-b^2)/2\,(a^2+b^2)$. The centre of the circle will lie on the chord common to the ellipse and its osculating circle at D, if this chord be perpendicular to the tangent to the former circle at D. The equation of the chord is $x/a\cos\delta-y/b\sin\delta-\cos2\delta=0$, and the condition of perpendicularity is

$$a/b\tan\delta=b/a\cot\delta\,(3a^2+b^2)/(3b^2+a^2)\ \text{ or }\ \tan\delta=b/a\left[(3a^2+b^2)/(3b^2+a^2)\right]^{\frac{1}{2}},$$

the value previously found.

10441. (Prof. Orchard, M.A.)—Give (1) an algebraical proof of the following result:—Having put down any sum of money (£. *s*. *d*.): (*a*) Reverse (so that pence come under pounds), and subtract; (*b*) Reverse this last sum, and add; then the result is constant, and is £12. 18*s*. 11*d*., as shown in the annexed example. Also (2) generalise the theorem.

£	*s*.	*d*.
10	17	5
5	17	10
4	19	7
7	19	4
12	18	11

Solution by Prof. Anderson, M.A.; Rev. H. Sewell, B.A.; *and others.*

Let x, y, z be the values in decreasing order of three coins, $x=ky$, $y=k'z$; and suppose k, k' both integers and $k>k'$; then any sum of money may be written $ax+by+cz$, where a, b, c are positive integers. When reversed, this becomes $cx+by+az$; and, if this is to express a sum of money in the ordinary way, we must have $a<k'$, and, to make the subtraction possible, we must assume $c<a$. Also b is less than k.

Subtracting, we get $(a-c)(x-z)=(a-c)\left[(k-1)\,y+(k'-1)\,z\right]$.

Let $a-c=k-q=k'-p$, where $q<k$ and $p<k'$; then the result of

subtraction is $(a-c-1)\,ky+qy+(a-c-1)\,k'z+pz$

$=(a-c-1)\,x+(a-c-1+q)\,y+pz=(k'-p-1)\,x+(k-1)\,y+pz.$

Reversing, we get $px+(k-1)\,y+(k'-p-1)\,z,$

and by addition $(k'-1)\,x+2(k-1)\,y+(k'-1)\,z$, or $k'x+(k-2)\,y+(k'-1)\,z.$

This is independent of a, b, c, and expresses a sum of money if $k \not< 2$, which is in accordance with the supposition $k > k' > 1$. If, as is the case in pounds, shillings, and pence, we make $k'12$ and $k20$, we get £12. 18*s.* 11*d.*

[A host of correspondents solve part (1) alone, thus:—Let $a > c$; then the annexed process explains itself:—

£.	*s.*	*d.*
a	*b*	*c*
c	*b*	*a*
$a-c-1$	19	$c+12-a$
$c+12-a$	19	$a-c-1$
11	38	11

Mr. Davis adds that this *trick*—as he calls it—has been "current in well-informed City-circles for some months. When performing it a second time it is advisable to say after the second reversal, 'Set down £2. 0*s.* 6*d.* under the last line, and add the three.' The result is, of course, £12 + 18*s.* 11*d.* + £2. 0*s.* 6*d.* = £14. 19*s.* 5*d.*, which calculation must be done mentally. Otherwise it will soon be suspected that the result (as above) is always the same."]

10336. (Maurice d'Ocagne.)—Les perpendiculaires élevées par le sommet C et par le milieu M du côté BC, à ce côté, coupent le côté AB respectivement en D et en P. La perpendiculaire élevée en C à CP coupe PM en Q. Démontrer que la droite DQ et la perpendiculaire élevée en C, à CA, se coupent sur la symédiane issue de A du triangle ABC.

Solution by Revs. T. R. Terry, M.A., *and* T. Galliers, M.A.

Let R be the intersection of DQ and the line through C perpendicular to CA; then equation

to CD is $\beta+\alpha\cos C=0,$

to PM is $\alpha\sin(B-C)+\beta\sin B-\gamma\sin C=0,$

to CP is $\alpha\sin(B-C)+\beta\sin B=0,$

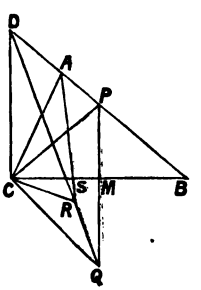
to CQ is $\alpha\cos(B-C)+\beta\cos B=0,$

to DQ is $\alpha\cos C+\beta-\gamma\sin B\sin C=0,$

to CR is $\alpha+\beta\cos C=0;$

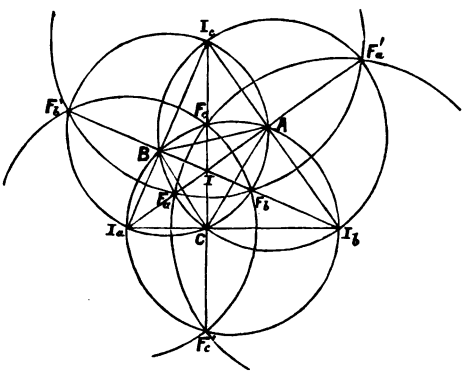
therefore the equation to AR is

$$\beta\sin C-\gamma\sin B=0.$$



10453. (Professor Gob.)– Soient F_a, F'_a les foyers d'une hyperbole tangente au côté BC d'un triangle ABC et asymptote aux côtés AB, AC. Soient F_b, F'_b et F_c, F'_c les foyers des deux autres hyperboles analogues. Démontrer que les quadrilatères $BF_a\,CF_a$, $F_b\,F_c\,F'_b\,F'_c$ sont harmoniques.

Solution by Professor Schoute.



In the diagram I, I_a, I_b, I_c are the centres of the circles touching the sides of ABC.

1. The foci F_a, F'_a are the point common to the bisector AI of the angle BAC and the circle BCI_bI_c. For the circles through F_a and F'_a cut AB in an involution BB′ of which A is the centre, and as AB′ = AC they meet the semi-lines AB, AC in points B, C, for which AB . AC = constant; this proves that BC envelopes the hyperbola with the asymptotes AB, AC and the foci F_a, F'_a.

2. The diagonals BC, $F_aF'_a$ of the inscribed quadrangle $BF_a\,CF'_a$ are conjugated with respect to the circle that passes through its vertices; in other words, the quadrangle $BF_a\,CF'_a$ is harmonical. For the complete quadrilateral IBI_aC shows that the point common to BC and I_bI_c is separated harmonically from A by I_b and I_c, and this proves that this pole of $F_aF'_a$ is situated on BC.

3. The points F_b, F'_b, F_c, F'_c are situated on a circle of which A is the centre. For, as $I_aB\,.\,I_aI_c = I_aC\,.\,I_aI_b$, we find $I_aF_b = I_aF_c$.

4. The quadrangle $F_bF_cF'_bF'_c$ is harmonical. For, as F_b is on the circle of which I_aI_c is a diameter, I_c is the pole of $F_bF'_b$ with respect to the circle (I_a), and this point lies on $F_cF'_c$, &c.

10400. (D. Biddle.)—Three equal spheres touch each other. Find the probability that the plane which cuts one cuts all.

Solution by Professor Syamadas Mukhopâdhyây, M.A.

The chance of a random plane's intersecting a convex surface is represented by $\mu \int p\rho^{-2}\,ds$, where ds is an element of surface, ρ the specific

curvature at the element, p the length of perpendicular from any origin within the figure to the tangent plane at the element, and μ a constant. The chance, therefore, of a random plane's intersecting a sphere of radius R is $4\mu\pi R$.

The chance of a random plane's intersecting a system of two equal spheres in contact is easily seen to be

$$4\mu\pi R + \mu\int_0^{\frac{1}{2}\pi} 2R\cos\theta\, 2\pi\sin\theta\, d\theta = 6\mu\pi R.$$

The chance of a random plane's intersecting a system of three equal spheres in contact is seen to be

$$3\int_{-\frac{1}{6}\pi}^{\frac{1}{6}\pi}\int_{-\frac{1}{6}\pi}^{\frac{1}{6}\pi} 2R\,(\cos\theta\cos\phi + 1)\, d\theta\cos\theta\, d\phi = 7\mu\pi R.$$

Let p_1, p_2, p_3 represent the chances of a random plane's intersecting only one, two, and three spheres, respectively, of the last system; then

$$4\mu\pi R = p_1 + 2p_2 + p_3, \quad 6\mu\pi R = p_2 + p_3 + 2(p_1 + p_2),$$
$$7\mu\pi R = 3p_1 + 3p_2 + p_3, \quad \therefore\ p_1 = p_2 = p_3 = \mu\pi R.$$

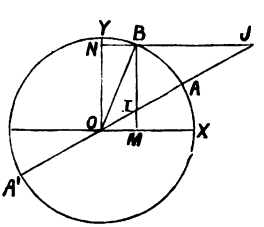
The probability of the present question is therefore equal to

$$\mu\pi R/(4\mu\pi R) = \tfrac{1}{4}.$$

10414. (Professor DE LONGCHAMPS.)—Soient OX, OY deux rayons rectangulaires dans un cercle Δ. Par O, on trace deux rayons OA, OB, symétriques par rapport à la bissectrice de YOX. Par B, on mène des parallèles aux droites OX, OY; elles rencontrent OA aux points I, J. Démontrer que la polaire de I, relativement à Δ, passe par J.

Solution by R. WHAPHAM, B.A.; D. T. GRIFFITHS, M.A.; *and others.*

$\angle IJB = IOM = BON = OBI$;
therefore $\angle OIB = OBJ$;
therefore ΔOIB is similar to OBJ;
therefore $OI \cdot OJ = OB^2 = OA^2$;
therefore I and J are harmonic conjugates with respect to A and A'; therefore the polar of I passes through J.



10456. (The EDITOR.)—In the base BC of a triangle find a point P, such that $AP^2 = m \cdot BP \cdot CP$.

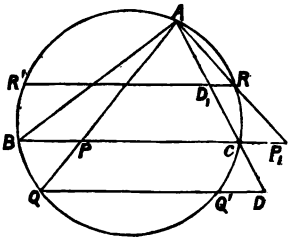
Solution by J. C. St. Claie; J. J. Barniville; *and others.*

On AC take points D, D_1, so that

$$CA = m \cdot CD = m \cdot CD_1;$$

and through D, D_1 draw parallels to BC cutting the circle ABC in QQ', RR'; then, if these points be joined to A, the joining lines will intersect BC in two internal points P, and two external points P_1, all of which fulfil the required condition. For we have

$$AC = m \cdot CD, \qquad AP = m \cdot PQ;$$



therefore $$AP^2 = m \cdot AP \cdot PQ = m \cdot BP \cdot CP.$$

If $m > 1$, D_1 lies within AC, and there will be two external points P_1. With this exception, the parallels to BC through D, D_1 may or may not meet the circle.

10116. (Professor Emmerich, Ph.D.)—P, Q being the foci of an in-conic of the triangle ABC, D its point of contact with BC, prove the proportion $$BD : CD = b \cdot BP \cdot BQ : c \cdot CP \cdot CQ.$$

Solution by Professor Nash, M.A.

The equation of an in-conic having a focus at the point $\alpha\beta\gamma$ may be written $$\left[x\alpha\,(\beta^2+\gamma^2+2\beta\gamma\cos A)\right]^{\frac{1}{2}} + \ldots = 0.$$

If α', β', γ' be the coordinates of the other focus Q, $\alpha' = \dfrac{2\Delta\beta\gamma}{\Sigma a\beta\gamma}$, $\beta' =$ &c.;

$$PA^2 \sin^2 A = \beta^2+\gamma^2+2\beta\gamma\cos A,$$

$$PA'^2 \sin^2 A = a^2(\beta^2+\gamma^2+2\beta\gamma\cos A)/(\Sigma a\beta\gamma)^2;$$

therefore $$PA \cdot PA' \sin^2 A = a(\beta^2+\gamma^2+2\beta\gamma\cos A)/(\Sigma a\beta\gamma).$$

Hence the equation of the conic can be written

$$ax\,(AP \cdot AQ)^{\frac{1}{2}} + by\,(BP \cdot BQ)^{\frac{1}{2}} + cz\,(CP \cdot CQ)^{\frac{1}{2}} = 0;$$

therefore, &c.

10467 & 10504. (Walter Stott.)—Construct a pentagon when there are given the mid-points of (1) the diagonals or (2) the sides.

Solution by R. F. Davis, M.A.; Professor Schoute; *and others.*

(10467.) Let PQRST be a pentagon; p, q, r, s, t the mid-points of the diagonals PR, QS, RT, SP, TQ. Let G be the centroid of five equal masses placed at the angular points of PQRST; then G is the same for

$pqrst$, and is therefore known. Let o be the mid-point of pq; join oG and produce it to T, so that $GT = 4Go$. This determines T; whence, &c.

(10504.) The same method applies to this form of the problem.

[Putting $v_1 = \frac{1}{5}(u_5+u_2)$, $v_2 = \frac{1}{5}(u_1+u_3)$, ... $v_5 = \frac{1}{5}(u_4+u_1)$;

$w_1 = \frac{1}{5}(v_3+v_4)$, $w_2 = \frac{1}{5}(v_4+v_5)$, ... $w_5 = \frac{1}{5}(v_2+v_3)$,

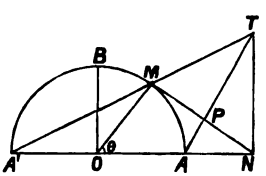
we have $\Sigma v = \Sigma u$, $4w_k + u_k = \Sigma u$ $(k = 1, 2, 3, 4, 5)$ ..... (1, 2).

If $u_k = x_k + iy_k$ be the complex coordinates of the vertex A_k of $A_1A_2A_3A_4A_5$, the pentagon to be constructed, v_k will denote the vertex B_k of the given pentagon $B_1B_2B_3B_4B_5$ formed by the mid-points of the diagonals A_5A_2, &c., and w_k the mid-point C_k of the side $B_{k+2}B_{k+3}$ of the given pentagon. Now the relation (1) shows that the centroid G of five equal masses at B_1, B_2, B_3, B_4, B_5 is also the centroid of five equal masses at A_1, A_2, A_3, A_4, A_5. And (2) shows that the points A_k and C_k are collinear with G and $A_kG = 4GC_k$, &c. We can thus determine G by means of the given points B, &c.]

10398. (Capitaine DE ROCQUIGNY.)—En un point quelconque M d'un quadrant de cercle AB, on mène une tangent, rencontrant le rayon OA en N. On élève sur OA la perpendiculaire NT = NM, et l'on joint AT, MT. Démontrer que (1) MT passe par un point fixe; (2) le lieu de T est une hyperbole équilatère (théorème connu); (3) le point d'intersection des droites NM et AT se meut sur une hyperbole.

Solution by Rev. T. GALLIERS, M.A.; R. KNOWLES, B.A.; *and others.*

1. Let $OA = a$, $\angle AOM = \theta$, AOA′ a diameter; then the equation of MT will be $y\cos\frac{1}{2}\theta - (x+a)\sin\frac{1}{2}\theta = 0$, and this clearly passes through the point $(-a, 0)$, that is, A′.



2. Since $TN^2 = MN^2 = ON^2 - OM^2$, the equation of the locus of T will be $x^2 - y^2 = a^2$, an equilateral hyperbola.

3. The equations of MN, AT are $x\cos\theta + y\sin\theta = a$, $y = \cot\frac{1}{2}\theta\,(x-a)$; hence the equation of the locus of P is $x^2 - 3y^2 = a^2$, an hyperbola, of which the semi-axes are a, $a/(\sqrt{3})$.

9051. (J. L. MACKENZIE, M.A.)—Through a point O in the plane of two conics U and V, a line is drawn cutting one conic in A and the other in B. Prove that the locus of a point P which divides AB in the ratio $\lambda : \mu$ $(\lambda + \mu = 1)$ is in general a curve of the eighth order. If U and V be similar, and similarly situated, the locus reduces to a curve of the sixth order passing through the intersections of the conics $U_2 + \lambda U_1 + \lambda^2 U_0 = 0$ and $\lambda^2(U_1^2 - 4U_0U_2) - \lambda\mu V_1(U_1 + 2\lambda U_0) + \mu^2 V_0 U_2 = 0$

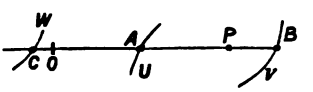
(where U_2 denotes terms of second degree in the equation of U, &c.), and also through the intersections of $V_2+\mu V_1+\mu^2 V_0=0$,

and $$\mu^2(V_1^2-4V_0V_2)-\lambda\mu U_1(V_1+2\mu V_0)+\lambda^2U_0V_2=0.$$

Solution by Professor SCHOUTE.

1. The condition of the problem is

$$OP=\lambda OA+\mu OB.$$



As any line l through O meets either of the conics U, V in two points, this line contains four segments AB; hence any line l determines four points P of the locus. This proves that the order of the locus exceeds by four the order of multiplicity of O on the curve. Now OP is zero, when $OB=-\lambda/\mu\ OA$. But the locus of C, determined by the equation $OC=-\lambda/\mu\ OA$, is the conic W obtained by multiplying the radii vectores of U with $-\lambda/\mu$. As this conic meets V in four points, the locus is a curve of the eighth order, that passes four times through O. The tangents at O are the four lines passing through the points common to V and W.

2. We verify this result by analysis. When the substitution $x=r\cos\phi$, $y=r\sin\phi$, in the given equations of U and V, gives

$$r^2u_2+ru_1+u_0=0,\quad r^2v_2+rv_1+v_0=0,$$

where u_k and v_k represent homogeneous functions of the order k in $\cos\phi$ and $\sin\phi$, the equation of the locus is found by eliminating ρ_1 and ρ_2 between $\rho=\lambda\rho_1+\mu\rho_2$, $\rho_1^2u_2+\rho_1u_1+u_0=0$, $\rho_2^2v_2+\rho_2v_1+v_0=0$.

So we find $$\rho=\frac{-\lambda u_1}{2u_2}-\frac{\mu v_1}{2v_2}\pm\frac{\lambda}{2u_2}\sqrt{u_1^2-4u_2u_0}\pm\frac{\mu}{2v_2}\sqrt{v_1^2-4v_2v_0};$$

or, as $$\rho^k u_k=U_k\quad\text{and}\quad\rho^k v_k=V_k,$$

$$\left\{\left(1+\frac{\lambda U_1}{2U_2}+\frac{\mu V_1}{2V_2}\right)^2-\frac{\lambda^2}{4U_2^2}(U_1^2-4U_2U_0)-\frac{\mu^2}{4V_2}(V_1^2-4V_2V_0)\right\}^2$$
$$=\frac{\lambda^2\mu^2}{2U_2\cdot V_2^2}(U_1^2-4U_2U_0)(V_1^2-4V_2V_0).$$

If we put $$U_2+\lambda U_1+\lambda^2U_0\equiv U',\quad V_2+\mu V_1+\mu^2V_0\equiv V',$$

this equation becomes

$$(U'V_2+U_2V'-U_2V_2)^2+\lambda\mu U_1V_1(U'V_2+U_2V'-U_2V_2)$$
$$+\lambda^2\mu^2(U_2U_0V_1^2+U_1^2V_2V_0-4U_2U_0V_2V_0)=0,$$

which proves that the locus is of the eighth order and passes four times through O. The tangents at this quadruple point are represented by

$$(\lambda^2U_0V_2+\mu^2U_2V_0)^2+\lambda\mu U_1V_1(\lambda^2U_0V_2+\mu^2U_2V_0)$$
$$+\lambda^2\mu^2(U_2U_0V_1^2+U_1^2V_2V_0-4U_2U_0V_2V_0)=0,$$

which can be thrown into the form

$$\begin{vmatrix} \mu^2U_2, & -\lambda\mu U_1, & \lambda^2U_0, & 0 \\ 0, & \mu^2U_2, & -\lambda\mu U_1, & \lambda^2U_0 \\ V_2, & V_1, & V_0, & 0 \\ 0, & V_2, & V_1, & V_0 \end{vmatrix} = 0,$$

of the equation of the lines passing through O and one of the points common to $\mu^2U_2 - \lambda\mu U_1 + \lambda^2U_0 = 0$, $V_2 + V_1 + V_0 = 0$,

i.e., common to the conics V and W.

3. If U and V are similar and similarly situated, the locus degenerates into the two lines through O parallel to the asymptotes of U and V and into a curve of the sixth order, of which O is a double point. In this case two of the four points common to V and W lie at infinity.

In this case $V_2 = U_2$. Then the equation contains the factor U_2. Omitting this factor we find the equation of the sextic in the form

$$U_2(U' + \mu V_1 + \mu^2 V_0)^2 + \lambda\mu U_1 V_1 (U' + \mu V_1 + \mu^2 V_0) + \lambda^2\mu^2 (U_0 V_1^2 + U_1^2 V_0 - 4U_2 U_0 V_0) = 0.$$

For $U' = 0$, we find

$$U_2(V_1 + \mu V_0)^2 + \lambda U_1 V_1 (V_1 + \mu V_0) + \lambda^2 (U_0 V_1^2 + U_1^2 V_0 - 4U_2 U_0 V_0) = 0,$$

and, as the coefficient of V_1^2 vanishes, by dividing out V_0, we have

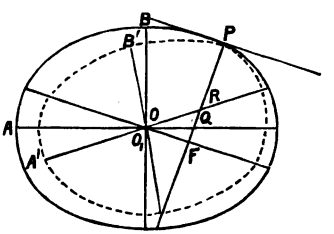
$$\lambda^2 (U_1^2 - 4U_2U_0) + \mu V_1 (\lambda U_1 + 2U_2) + \mu^2 U_2 V_0 = 0,$$

or

$$\lambda^2 (U_1^2 - 4U_2U_0) - \lambda\mu V_1 (U_1 + 2\lambda U_0) + \mu^2 U_2 V_0 = 0, \text{ \&c.}$$

10286. (R. Knowles, B.A.)—Two given conics osculate at a point P; their transverse axes meet the normal at P in Q and R respectively; prove that the ratio QP : RP is constant for all positions of P.

Solution by C. Bickerdike; A. J. Pressland; *and others.*



Here $PF \cdot PQ = BO^2$,

$PF \cdot PR = B'O^2$;

hence

$PF \cdot PQ : PF \cdot PR = BO^2 : B'O^2$

= constant,

therefore $PQ : PR$ = constant.

10031. (Professor Déprez.)—Si la droite OH qui joint le centre du cercle circonscrit au triangle ABC, à l'orthocentre, est perpendiculaire à la bissectrice de l'angle ACB, l'angle C est nécessairement égal 60°.

Solution by Prof. BEYENS; R. KNOWLES, B.A.; *and others.*

Soit CC_2 bissectrice de l'angle C, elle sera aussi de l'angle OCH, parce que

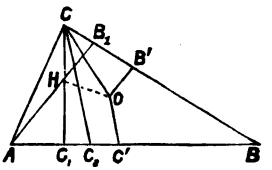
$$OCB' = 90° - A, \quad ACH = 90° - A.$$

Mais $CH = 2C'O$, C' étant milieu de AB,

$$CH = 2C'O = 2OA \cos AOC' = 2OC \cos C;$$

mais CC_2, bissectrice de l'angle COH. étant perpendiculaire au côté OH, on aura

$$\cos C = \tfrac{1}{2} = \cos 60°.$$



10387. (The EDITOR.)—Solve the equations

$$x^2(y-z) = a, \quad y^2(z-x) = b, \quad z^2(x-y) = c.$$

Solution by D. BIDDLE; WALTER STOTT; *and others.*

We have $x^2y^2z^2 = -abc/(a+b+c)$, $a/x^2 + b/y^2 + c/z^2 = 0$;

therefore $$ay^2z^2 + bx^2z^2 + cx^2y^2 = 0;$$

$$(y-z)\frac{-abc}{a+b+c} + bx^2z^2 + (x-y)\frac{-abc}{a+b+c} = 0; \quad \therefore \; bx^2z^2 = \frac{(x-z)\,abc}{a+b+c};$$

therefore $$z = \frac{\left\{a^2c^2 + 4(a+b+c)\,acx^3\right\}^{\frac{1}{2}} - ac}{2(a+b+c)\,x^2};$$

similarly, $$y = \frac{\left\{a^2b^2 - 4(a+b+c)\,abx^3\right\}^{\frac{1}{2}} + ab}{2(a+b+c)\,x^2}.$$

The first equation in the question then gives us

$$x = \left[\frac{a^2}{(b+c)^2}\left\{c - b \pm \left(-4bc - \frac{bc(b+c)^2}{a(a+b+c)}\right)^{\frac{1}{2}}\right\}\right],$$

and y and z follow. But, in order that real roots may be obtained, one of the known quantities must differ in sign from the other two, and be (as a neutral number) less than their sum.

10313. (Professor DE LONGCHAMPS.)—Soient A un point quelconque pris dans le plan d'une ellipse E; B, C les points de contact des tangentes issues de A. (1) Démontrer que l'orthocentre du triangle ABC et le point A sont conjugués harmoniques par rapport au cercle de Monge (cercle circonscrit au rectangle des axes). Examiner le cas particulier où le point A est situé sur E, et démontrer le théorème suivant:—La normale en un point A d'une ellipse E rencontrant le cercle de Monge aux points P et Q, le centre de courbure correspondant est le symétrique,

par rapport à A, du conjugué harmonique de ce point, relativement au segment PQ.

Solution by Rev. T. R. Terry, M.A.; Prof. Mukhopâdhyây; *and others.*

Let $\alpha+\beta$ and $\alpha-\beta$ be the eccentric angles of B and C. Then the coordinates of A are $a\cos\alpha\sec\beta$, $b\sin\alpha\sec\beta$.

Line through B perpendicular to AC is

$$ax\sin(\alpha-\beta)-by\cos(\alpha-\beta)=a^2\cos(\alpha+\beta)\sin(\alpha-\beta)-b^2\sin(\alpha+\beta)\cos(\alpha-\beta),$$

and line through C perpendicular to AB is

$$ax\sin(\alpha+\beta)-by\cos(\alpha+\beta)=a^2\cos(\alpha-\beta)\sin(\alpha+\beta)-b^2\sin(\alpha-\beta)\cos(\alpha+\beta);$$

therefore, subtracting, we see that the orthocentre lies on

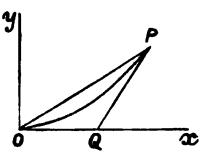
$$ax\cos\alpha+by\sin\alpha=(a^2+b^2)\cos\beta,$$

that is, lies on the polar of A with respect to Monge's circle—whence (1).

In the limiting case the orthocentre clearly lies on the normal at A, therefore (2) follows from (1) by means of the known theorem that the orthocentre of a triangle, formed by QO, QP, the tangents to any curve, and OP the chord of contact, in the limit lies on the normal produced outwards to a distance equal to the radius of curvature.

[Of the theorem cited above the following is a very simple proof:—

Take OQ as axis of x, and let $OP=s$; then the coordinates of P are $(s, s^2/2\rho)$, and of Q $(\frac{1}{2}s, 0)$; also the line through P perpendicular to OQ is $x=s$, and the line through Q perpendicular to OP is $s(x-\frac{1}{2}s)+y(s^2/2\rho)=0$; therefore at the orthocentre $y=-\rho$.]



10393. (Rev. G. Richardson.)—If the transversals AD, BE, CF of the triangle ABC divide the opposite sides in the ratios $l:1$, $m:1$, $n:1$, respectively, prove that (1) the area of the triangle they form : area ABC

$$=(lmn-1)^2:(mn+m+1)(nl+n+1)(lm+l+1);$$

(2) the two triangles are similar if

$$(l+1)(mn+m+1),\ (m+1)(nl+n+1),\ (n+1)(lm+l+1)$$

are in the duplicate ratios of the sides of ABC; and (3) if any one of the transversals is trisected by the other two, $4lmn=1$.

Solution by J. J. Barniville; Professor Beyens; *and others.*

1. Let the triangle be PQR; and let AP meet BC in D′. Then, by the theory of transversals, $D'C=mn\,.\,D'B$, and

$$\Delta PQR:ABC=(RP:BE)(QR:RA)(AE:AC);$$

$BR : BE = lm + l : lm + l + 1$, $BP : BE = m + 1 : mn + m + 1$;
hence $RP : BE = (m+1)(lmn-1) : (lm+l+1)(mn+m+1)$.
Also $QR : RA = lmn - 1 : nl + n + 1$, $AE : AC = 1 : m + 1$;
hence $PQR : ABC = (lmn-1)^2 : (lm+l+1)(mn+m+1)(nl+n+1)$.

2. Also, if
$(l+1)(mn+m+1) : (m+1)(nl+n+1) : (n+1)(lm+l+1) = AB^2 : BC^2 : CA^2$,
then $$\frac{PQ}{CF} : \frac{QR}{AD} : \frac{RP}{BE} = AB^2 : BC^2 : CA^2;$$
hence the triangles are similar.

3. The anharmonic pencil $A(BDD'C) = A(BRPE)$. Hence
$$lmn = \frac{BD}{DC} \cdot \frac{D'C}{D'B} = \frac{1}{4}.$$

10243 & 10046. (Maurice d'Ocagne.)—Soit A un point arbitrairement choisi sur une hyperbole équilatère, et soient B et C les extrémités d'un diamètre quelconque de cette courbe. La tangente en A à cette hyperbole équilatère est symédiane du triangle ABC. (Ce théorème permet, connaissant le centre et deux points d'une hyperbole équilatère, de construire les tangentes en ces points.) *Corollaire.*—Si un cercle et une hyperbole équilatère sont concentriques, les tangentes à l'hyperbole aux extrémités d'un de leurs diamètres communs sont perpendiculaires à l'autre diamètre commun. [For Quest. 10046, see Vol. LII., p. 83.]

Solution by John J. Barniville.

10046 (1) and 10243. Let ABC be (h, k), $(-x', -y')(x', y')$, and let the tangent be AD; then $1 + \tan BAO = a^2/(hx' + ky') = 1 + \tan CAD$.

10046 (2). By examining the coefficients, we see that the line through the intersection of $y'x + x'y = 0$ and $(y'-k)x - (x'-h)y + x'k - y'h = 0$, perpendicular to the latter, is identical with the line through the intersection of $x'x + y'y = 0$ and $(y'+k)x - (x'+h)y + x'k - y'h = 0$, perpendicular to the same line.

10046 (3) and 10243, Cor. The tangent at $x'y'$ is $xx' - yy' = a^2$, and the diameter through $(x', -y')$ and $(-x', y')$ is $xy' - yx' = 0$.

10038. (Professor Bordage.)—Show that the trilinear equation
$$\alpha(\beta-\gamma)^3 + \beta(\gamma-\alpha)^3 + \gamma(\alpha-\beta)^3 = 0,$$
represents four straight lines.

Solution by J. J. Barniville; L. Wiener; *and others.*

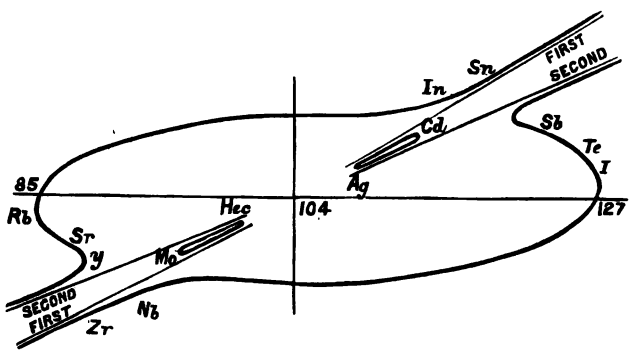
Factorising, we have $(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma) = 0$, which represents four right lines, three of them being bisectors of the angle.

10440. (Professor HAUGHTON, F.R.S.) — Describe the quartic curve passing through the following points, whose Cartesian coordinates (with $x = 104$ as origin) are (1) Rubidium (85, −1) (2) Strontium (87, −2); (3) Yttrium (89, −3); (4) Zirconium (90, −4); (5) Niobium (94, −3); (6) Molybdenum (96, −2); (7) Hecaton (100, −1); (8) Silver (108, 1); (9) Cadmium (112, 2); (10) Indium (113, 3); (11) Tin (118, 4); (12) Antimony (121, 3); (13) Tellurium (125, 2) (14) Iodine (127, 1).

Solution by GEORGE GOLDTHORPE STORR, M.A.

Let the equation of the quartic curve be

$$A_1y^4 + A_2y^3x + A_3y^2x^2 + A_4yx^3 + x^4 + B_1y^3 + B_2y^2x + B_3yx^2 + B_4x^3 + C_1y^2 + C_2yx + C_3x^2 + D_1y + D_2x + E_1 = 0 \quad \ldots\ldots\ldots (1)$$



Since molybdenum, hecaton, silver, and cadmium are on the straight line $x = 4y$, we have

$$B_1 + 4B_2 + 16B_3 + 64B_4 = 0, \quad D_1 + 4D_2 = 0,$$
$$C_1 + 4C_2 + 16C_3 + 4\phi = 0, \quad E_1 - 4\phi = 0,$$

where $\phi = A_1 + 4A_2 + 16A_3 + 64A_4 + 256$.

By substituting in (1) the values of the various coordinates, we obtain ten other equations for calculating A_1, A_2, &c. These are found to be

$A_1 = 729{\cdot}645$,	$B_1 = 104{\cdot}336$,	$C_1 = -6583{\cdot}969$,	$D_1 = -97{\cdot}68$,
$E_1 = -51{\cdot}868$,	$A_2 = -409{\cdot}429$,	$B_2 = -110{\cdot}584$,	$C_2 = 3348{\cdot}429$,
$D_2 = 24{\cdot}420$,	$A_3 = 83{\cdot}048$,	$B_3 = 37{\cdot}248$,	$C_3 = -421{\cdot}557$,
	$A_4 = -10{\cdot}775$,	$B_4 = -4{\cdot}031$.	

The curve has two real asymptotes represented by the equation

$$y = mx + n,$$

where $m = {\cdot}2776$, $n = -{\cdot}00635$ for the first, and $m = {\cdot}223$, $n = -{\cdot}0393$ for the second. The curve consists of two real ovals lying inside the acute angles between the asymptotes and two infinite branches occupying the obtuse angles of the asymptotes.

[This solution has been derived from the Proposer's valuable and interesting Paper, entitled *Geometrical Illustrations of Newlands' and Mendelejeff's Periodic Law of the Atomic Weights of the Chemical Elements*, read before the Royal Irish Academy on April 28, 1888.]

10476. (Professor SYLVESTER.)—Prove (1) that it is always possible to transform $ax\xi + bx\eta + cx\zeta + a'y\xi + b'y\eta + c'y\zeta + a''z\xi + b''z\eta + c''z\zeta$, where all the coefficients are real, into the form $Ax\xi + By\eta + Cz\zeta$, by means of two *real* orthogonal linear substitutions imposed on x, y, z and ξ, η, respectively; and show (2) how to construct the equation for determining A, B, C.

Solution by W. J. JOHNSTON, M.A.; Prof. CHAKRAVARTI; *and others.*

The application of the contravariant operator $\left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 + \left(\frac{d}{dz}\right)^2$ to the square of the function gives

$$(a\xi + b\eta + c\zeta)^2 + \ldots + \ldots = A^2\xi'^2 + B^2\eta'^2 + C^2\zeta'^2 \ (\equiv u).$$

By the law of inertia A^2, B^2, C^2 are positive, and ξ, η, ζ real functions of ξ', η', ζ'.

It is the same problem to find the principal planes of the cone $u = 0$. A^2, B^2, C^2 are the roots of the equation in λ,

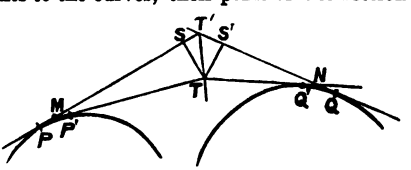
$$\text{Discriminant } \{u - \lambda(\xi^2 + \eta^2 + \zeta^2)\} = 0.$$

If we substitute in $ax\xi + \&c. = Ax'\xi' + \&c.$ the expressions for ξ, η, ζ, and compare coefficients of ξ', η', ζ'; then, as the coefficients a, b, &c. are all real, we see that x, y, z are real linear functions of x', y', z'.

10448. (Professor NEUBERG.)—Deux droites d, d' roulent sur deux courbes données Δ, Δ' avec des vitesses angulaires, qui sont constamment égales et de sens contraire. Trouver la tangente à la courbe décrite par le point d'intersection de ces droites.

Solution by Professors ANDERSON, SARKAR, *and* SCHOUTE.

Let PT, QT be the tangents to the curves, their point of intersection being T. Draw the tangents P'T, Q'T, making small equal angles θ with these, and intersecting in T'. Let these intersect the other pair in the points M, N; and draw T'S, T'S' perpendicular to PT and QT; then



$$T'T \sin \angle T'TS = T'S = MT'.\theta, \quad T'T \sin T'TS' = T'S' = NT'.\theta.$$

Hence, in the limit, when θ becomes indefinitely small, and TT' becomes the tangent to the locus of T, we have

$$PT/QT = \sin(PTT')/(\sin(QTT');$$

thus the tangent at T to the locus of T is the symmedian through T of the triangle TPQ.

10378. (Professor WOLSTENHOLME, M.A., Sc.D.)—From a point O are drawn straight lines OP, OP' touching a given parabola in P, P'; find (1) the locus of O when the ratio OP : OP' is given; also prove that (2) this locus cuts the directrix at right angles in two points; (3) if from either of these points O the tangents OP, OP' meet the latus rectum in L, L', and LQ, L'Q' be also tangents in Q, Q' to the parabola,

$$OP : OP' = LP : LQ = L'Q' : L'P';$$

and (4) the focal distance of any point on the locus will be the least possible when the point lies on the tangent at the vertex of the parabola (being the point of contact of a common tangent to the locus and to the parabola), and this minimum distance bears to the distance between focus and vertex the given ratio of the tangents.

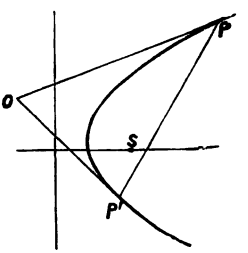
Solution by R. KNOWLES, B.A.; Rev. J. L. KITCHIN; *and others.*

(1) Let $y^2 = 4ax$ be the equation to the parabola, (xy) the point O; we have then $OP = (y^2-4ax)^{\frac{1}{2}}\left[\left\{(y^2-4ax)^{\frac{1}{2}}+y\right\}^2+4a^2\right]^{\frac{1}{2}}/2a$;

$$OP' = (y^2-4ax)^{\frac{1}{2}}\left[\left\{(y^2-4ax)^{\frac{1}{2}}-y\right\}^2+4a^2\right]^{\frac{1}{2}}/2a;$$

and if $OP : OP' = 1 : m$, the locus is

$$a^2(1-m^2)^2\left\{y^2+(x-a)^2\right\} - m^2y^2(y^2-4ax) = 0.$$



(2) This meets the directrix in two points

$$-a, \quad \pm a(1-m^2)/m;$$

$$dy/dx = \left\{2a^2(1-m^2)^2(x-a)+4am^2y^2\right\} / \left\{2a^2(1-m^2)^2y-4m^2y^3+8am^2xy\right\} = 0,$$

when x, y are the points on the directrix.

(3) When O is one of these points, the coordinates of P, L, Q are, respectively,

a/m^2, $2a/m$; a, $a(1+m^2)/m$; am^2, $2am$;

therefore $LP : LQ = 1 : m = OP : OP'$;

similarly for L'P' and L'Q'.

(4) The focal distance of any point (xy)

$$= \left\{y^2+(x-a)^2\right\}^{\frac{1}{2}} = my(y^2-4ax)^{\frac{1}{2}}/a(1-m^2)$$

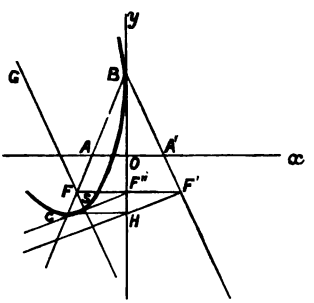
from the equation to the locus, and this is a minimum when $x = 0$, the equation to the tangent at the vertex; putting $x = 0$ in the locus, we have $y = a(1-m^2)^{\frac{1}{2}}/m$, and its distance from the focus $= a/m$;

$a/m : a = 1 : m$, the given ratio of the tangents.

8859. (N'Importe.)—On donne dans un plan deux axes de coordonnées rectangulaires Ox, Oy, et une droite AB définie par son coefficient angulaire m et son ordonnée à l'origine b, et l'on demande : (1) De trouver la direction des diamètres des paraboles tangentes à l'axe des y au point B où il est coupé par la droite AB et ayant leurs foyers sur cette dernière droite ; (2) D'écrire l'équation générale de ces courbes ; (3) De construire le lieu de leurs sommets ; (4) De construire le lieu des points où leurs tangentes sont parallèles à l'axe des x ; (5) De construire le lieu des pôles de l'axe des x relativement aux paraboles considérés.

Solution by Professor Schoute.

1. Let F on AB be the focus of one of the parabolas. As the tangent OB bisects the angle between radius vector BF through B and the diameter through B, the line BA′ (for which OA′ = AO) is the diameter through B. This proves that the parabolas have parallel axes. They form at the same time an ordinary and a tangential pencil.



2. When we consider the pencil as an ordinary pencil, it may be determined by the line BA′ counted twice and by the line OB combined to the line at infinity. Therefore the equation of the pencil is

$$(y-b+mx)^2+\mu x=0,$$

μ representing the parameter of the pencil.

3. When F is the focus, FG parallel to BA′ is the axis, F′ a point of the directrix, F″ a point of the tangent at the vertex, and S the vertex itself. Now, when F moves along BA, the form of the quadrangle BFSF″ does not change, and S describes the line BS.

4. The tangents passing through any point of the directrix are perpendicular to one another ; so the tangent parallel to Ox passes through H. Moreover, the polar of any point H of the directrix passes through the focus. Hence the locus of the point of contact C is the line BA.

5. As the parabolas form a tangential pencil, the locus of the pole of any line is a line. In our case the locus passes through A, A being the point of contact of Ox with the parabola, the directrix of which passes through O. And if the focus is in A, the pole of Ox through A is the point where the normal in A to Ox meets the directrix. So the locus of the pole is proved to be the line parallel to Oy through A.

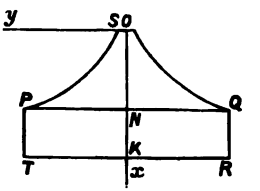
[The pole of Ox for the parabola with the focus F is on the parallel to FG that contains C ; and for this point P we have CP = CA.]

3395. (Professor Hudson, M.A.)—An indefinitely small piece of ice, the shape of which may be taken to be that of a right circular cylinder,

is floating with its axis vertical in water. The part immersed receives deposits of ice in such a manner as to continue cylindrical, the radius and axis receiving equal increments in equal times. Find the ultimate shape of the part not immersed. If the specific gravity of ice be ·96, prove that the surface is formed by the revolution of $y^2(9x-y)^{25} = a^{27}$.

Solution by Professor ANDERSON, M.A.

Take the highest point of the piece of ice as origin of coordinates, the axis of y being horizontal, and that of x vertical. Let the coordinates of P be x, y. Suppose PN increased by δy, and consequently NK by the same amount, the volume of PQRT will be increased by



$$\pi y^2 \delta y + 2\pi x y \delta y\,(y-x) = \pi(3y^2-2xy)\,\delta y.$$

The mass of the ice is thus increased by

$$\pi\rho\,(3y^2-2xy)\,\delta y,$$

being the specific gravity.

Hence the plane of floatation will be lowered in the ice by an amount δx given by $\pi(1-\rho)(3y^2-2xy)\,\delta y/\pi y^2$;

hence the differential equation of the curve is $y^2dx = \mu(3y^2-2xy)\,d$ where $\mu = 1-\rho$.

Let $x = vy$; the equation becomes $\dfrac{dy}{y} + \dfrac{dv}{v(2\mu+1)-3\mu} = 0$,

the solution of which is $y\left[(2\mu+1)v-3\mu\right]^{1/(2\mu+1)} = \text{constant}$.

Therefore $y\left[(2\mu+1)x/y-3\mu\right]^{1/(2\mu+1)} = \text{constant}$.

In the particular case $\mu = \frac{1}{25}$, and we get $y^2(9x-y)^{25} = a^{27}$, a being a constant.

If we make $x = 0$, we find $y = -a$, and therefore $OS = -a$, and consequently the constant a must be made indefinitely small. The solid thus approximates to a cone whose semi-vertical angle is $\tan^{-1} 9$.

10488. (Professor HENDRICKS.)—The *à priori* probability that *ace* will be thrown at a single cast with a die is $\frac{1}{6}$, $= k$ say. A, whose veracity is p_1, says that B, whose veracity is p_2, said that *ace* was thrown at a single cast; find the probability that *ace* was thrown. [This Question was proposed in our columns many years ago, but Professor HENDRICKS thinks that none of the solutions then published are satisfactory.]

Solution by D. BIDDLE.

A's assertion is twofold: he asserts that B made a statement regarding the event in question, and he further asserts that the statement was affirmative. Assuming, however, that B made a definite statement, affirming or denying the event from personal observation, the relative probabilities, as to his being true or untrue in *affirming* it, would be $kp_2 : (1-k)(1-p_2)$,

and in *denying* it, $(1-k)p_2 : k(1-p_2)$. If, therefore, we disregard the intervention of A, and suppose B to make his statement (whatever it may have been) direct to ourselves, we have the following probabilities:—

That B, when *affirming*, is right, $kp_2 / \{kp_2 + (1-k)(1-p_2)\}$ ..... (a),

,, ,, is wrong, $(1-k)(1-p_2) / \{kp_2 + (1-k)(1-p_2)\}$ (b),

That B, when *denying*, is right, $(1-k)p_2 / \{(1-k)p_2 + k(1-p_2)\}$ ... (c),

,, ,, is wrong, $k(1-p_2) / \{(1-k)p_2 + k(1-p_2)\}$ ... (d).

Now, on the supposition that B made a definite statement to A, the probability that *ace* was thrown is $p_1(a) + (1-p_1)(d)$. But A may have concocted the whole reference to B,

$$\therefore \quad P = p_1\{p_1(a) + (1-p_1)(d)\}$$
$$= kp_1\left\{\frac{p_1p_2}{kp_2 + (1-k)(1-p_2)} + \frac{(1-p_1)(1-p_2)}{(1-k)p_2 + k(1-p_2)}\right\}.$$

The probability that B *affirms* (or *denies*) forms no separate factor in P, depending for its estimation (according to the question) on the veracity or otherwise of A.

Examples.—1. Let $k = \frac{1}{8}$, $p_1 = \frac{6}{10}$, $p_2 = \frac{8}{10}$;

then $P_1 = \frac{3}{8}\frac{4}{5}\frac{4}{5} = \frac{4}{5}\frac{8}{8}\frac{2}{5} = \frac{8}{10}\left(\frac{5}{8}\frac{2}{8}\right)$.

2. Let $k = \frac{1}{8}$, $p_1 = \frac{8}{10}$, $p_2 = \frac{6}{10}$, in which p_1 and p_2 are transposed;

then $P_2 = \frac{4}{7}\frac{1}{0} = \frac{3}{5}\frac{2}{8}\frac{8}{0} = \frac{8}{10}\left(\frac{3}{5}\frac{2}{8}\right)$.

These examples show that, where the *à priori* probability is the same, a statement which has the better start has the less likelihood of being discredited. This is especially the case, as shown, where there is independent evidence that the supposed originator made a statement of some sort on the subject.

3. Let $k = \frac{1}{100000}$, $p_1 = \frac{8}{10}$, $p_2 = \frac{8}{10}$;

then $P_3 =$ nearly $\frac{6404}{8000008000}$, or about $\frac{8}{10}\left(\frac{1}{124000}\right)$.

This example shows that, where the *à priori* probability is small, a statement regarding any event has little chance of acceptance, when transmitted through a series of single witnesses, unless the veracity of these be beyond dispute (when, however, the fact is established, even according to the above formula). To render a statement of this nature highly probable, the evidence of a *number* of concurrent witnesses is usually requisite.

9336. (Professor Morley, M.A.)—In Salmon's *Curves*, § 216, we are told that in an acnodal cubic the acnode is the pole of the line through the three real inflexions with regard to the triangle formed by the three tangents. Here are meant the tangents at the inflexions. Prove that the proposition is also true if we take the tangents from the inflexions.

Solution by Professor A. M. Nash, M.A.

The equation of the cubic being $(x+y+z)^3 = 27xyz$, the acnode is the point $x = y = z$. The tangents from the inflexions are

$$x + 4y + 4z = 0, \quad 4x + y + 4z = 0, \quad 4x + 4y + z = 0.$$

The lines joining the intersection of the last two tangents to the acnode, and to the intersection of the first tangent with the line through the inflexions, *i.e.*, $x+y+z=0$, are $4x+y+4z \mp (4x+4y+z)=0$.

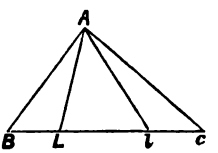
Therefore these lines are harmonic conjugates with respect to the two tangents; therefore, &c.

10374. (R. Tucker, M.A.)—The "N.P." circle of a triangle is the auxiliary circle of the in-ellipse $(a^2\cos A\alpha)^{\frac{1}{2}}+(b^2\cos B\beta)^{\frac{1}{2}}+(c^2\cos C\gamma)^{\frac{1}{2}}=0$, of which the foci are the circum- and ortho-centres. If L, M, N are the points of contact with the sides, prove that AL, BM, CN meet in the Isotomic of the circumcentre.

Solution by Rev. T. Gallibers, M.A.; G. G. Storr, M.A.; *and others.*

Let P, O be the ortho- and circum-centres; then the coordinates of P, O and the mid-point of OP are $2R\cos B\cos C$,

&c., $R\cos A$, &c., $\frac{1}{2}R\cos(B-C)$, &c. ..... (1).



Now, since the product of the perpendiculars from P, O on *any* side of the triangle

$= 2R^2\cos A\cos B\cos C$,

an in-ellipse can be described with P, O as foci; let its equation be $(L\alpha)^{\frac{1}{2}}+(M\beta)^{\frac{1}{2}}+(N\gamma)^{\frac{1}{2}}=0$;

then its centre (α, β, γ) is given by

$$\alpha : \beta : \gamma = Nb+Mc : Lc+Na : Ma+Lb \quad \ldots\ldots\ldots\ldots\ (3).$$

Putting $\alpha : \beta : \gamma = \cos(B-C) : \cos(C-A) : \cos(A-B)$, from (1), in (3), we have $L : M : N = a^2\cos A : b^2\cos B : c^2\cos C$.

Now the perpendicular from a focus of an ellipse on any tangent meets the tangent in a point lying on the auxiliary circle, whence it follows that the "N.P." circle of the triangle is the auxiliary circle of the ellipse. Let Al pass through O, then the last part of the question will be proved if we show that BL = Cl.

The equation of AL is, by (1), $\beta/(\sin^2 C\cos C)=\gamma/(\sin^2 B\cos B)$;

and at L, $\gamma/(\sin^2 B\cos B)=\{\beta/(\sin C)+\gamma/(\sin B)\}/(\sin B\cos B+\sin C\cos C)$

$=2a/(\sin 2B+\sin 2C)$;

hence $BL=\gamma/(\sin B)=2a\sin B\cos B/(\sin 2B+\sin 2C)$.

Again, the equation of Al is $\beta/(\cos B)=\gamma/(\cos C)$; and at L,

$\beta/(\cos B)=\{\beta/(\sin C)+\gamma/(\sin B)\}/\{\cos B/(\sin C)+\cos C/(\sin B)\}$

$=2a\sin B\sin C/(\sin 2B+\sin 2C)$;

$\therefore\ Cl=\beta/(\sin C)=2a\sin B\cos B/(\sin 2B+\sin 2C)$.

Hence BL = Cl; therefore AL, Al are isotomic conjugates with respect to the $\angle$BAC.

Similarly, if m, n be points on CA, AB such that Bm and Cn pass through O, we may show that BM, Bm are isotomic conjugates, as also CN, Cn. The straight lines AL, BM, CN obviously intersect in a point,

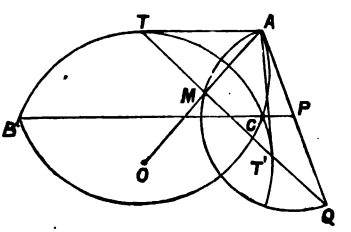
and this point, from what has preceded, is seen to be the isotomic conjugate of O with respect to the triangle ABC.

[If α, β, γ; α', β', γ' are points such that one is the isotomic conjugate of the other, then $a^2\alpha\alpha' = b^2\beta\beta' = c^2\gamma\gamma'$; therefore, if (α, β, γ) is the circumcentre, $\alpha'\beta'\gamma'$ is $1/a^2 \cos A$, $1/b^2 \cos B$, $1/c^2 \cos C$. The ellipse in the question is the envelope of the circumcircles of the *Isoscelian Hexagrams* of a triangle. See a paper by the PROPOSER in Vol. XXI. of the *Proc. Lond. Math. Society*, Nos. 370, 371, pp. 13, 26.]

10474. (J. MORRAU.)—De l'un des sommets d'un triangle, on mène des tangentes à un cercle quelconque passant par les deux autres sommets. Trouver le lieu du point où la corde de contact rencontre le diamètre qui passe par le premier sommet.

Solution by J. C. ST. CLAIR; R. F. DAVIS, M.A.; *and others.*

Let ABC be the given triangle. Draw AP tangent to the circumcircle ABC, meeting BC in P; and, with P as centre and radius PA, describe a circle cutting AP again in Q. Then this circle cuts any circle through B, C orthogonally, and consequently A and Q are conjugate points with respect to every circle of the co-axal system. Therefore all the chords of contact TT′ (being polars of A) pass through Q; and, since AMQ is a right angle, the locus of M is the circle upon AQ.



10499. (W. J. C. SHARP, M.A.)—A circle is just immersed in a fluid the density of which is known to vary as some function of the depth. It is found that the centre of pressure divides the vertical diameter into two parts, which are as 7 : 3; find the law of density.

Solution by D. EDWARDES, B.A.; Prof. ANDERSON, M.A.; *and others.*

Let the pressure at depth z be $z\phi(z)$; then

$$\int_0^d z^2\phi(z)(dz-z^2)^{\frac{1}{2}}\,dz \Big/ \int_0^d z\phi(z)\,(dz-z^2)^{\frac{1}{2}}\,dz = 7d/10,$$

d being the length of the diameter.

Differentiating with respect to d, and integrating by parts, we have

$$\int_0^d z^{\frac{3}{2}}(d-z)^{\frac{1}{2}}\left\{z\phi'(z)-\phi(z)\right\}dz = 0.$$

Now $\phi(z)$ is independent of d, and the last equation is to hold for all values of d, which requires that

$$z\phi'(z) - \phi(z) = 0;$$

therefore $\phi(z) = Az$, and $p = Az^2$. Then $g\rho = dp/dz = 2Az$, and the density varies directly as the depth. To verify, we have, at once,

$$d/dy\left\{y^2(y-y^2)^{\frac{3}{2}}\right\} = \tfrac{7}{2}y^2(y-y^2)^{\frac{1}{2}} - 5y^3(y-y^2)^{\frac{1}{2}},$$

and therefore $$\int_0^1 y^3(y-y^2)^{\frac{1}{2}}dy \Big/ \int_0^1 y^2(y-y^2)^{\frac{1}{2}}dy = \tfrac{7}{16}.$$

10475. (Professor POULAIN.)—Dans un triangle, démontrer que la distance d du centre de gravité au point de Lemoine vérifie l'égalité

$$9d^2(\Sigma a^2)^2 = -\Sigma a^6 + 3\Sigma a^4b^2 - 15a^2b^2c^2.$$

Solution by R. KNOWLES, B.A.; Prof. ANDERSON, M.A.; *and others.*

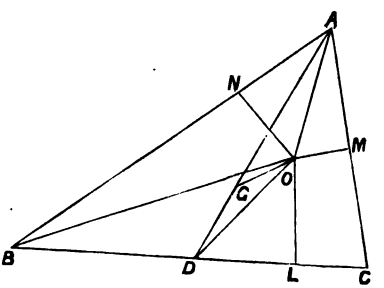
Let O be the Lemoine-point, G the centroid, and D the mid-point of BC. Then the co-ordinates of the centroid G and the Lemoine-point O are, respectively, $\Delta/3a$, &c.;

$$2a\Delta/(a^2+b^2+c^2), \text{ \&c.},$$

and we obtain

$$\begin{aligned} &9d^2(a^2+b^2+c^2)^2 \\ &= a^2(a^2+c^2-2b^2)(a^2+b^2-2c^2) \\ &+ b^2(a^2+b^2-2c^2)(b^2+c^2-2a^2) \\ &+ c^2(b^2+c^2-2a^2)(a^2+c^2-2b^2); \end{aligned}$$

hence, multiplying, we get the result stated.



10451. (Professor BUENGER.)—Démontrer qu'il existe deux points dont les distances aux sommets d'un tétraèdre donné ABCD soient proportionnelles à des nombres donnés α, β, γ, δ. Ces points divisent harmoniquement un diamètre de la sphère circonscrite au tétraèdre ABCD.

Solution by W. J. C. SHARP, M.A.; Prof. ANDERSON, M.A.; *and others.*

This proposition, which is true in space of any dimensions, may be proved as follows, using the notation of the solution of Questions 9093 and 9170 (Vol. XLVIII., p. 179). If $(\lambda, \mu, \nu \ldots)$ be the point, we have

$$\left(-S + V\frac{dS}{d\lambda}\right)\Big/\alpha^2 = \left(-S + V\frac{dS}{d\mu}\right)\Big/\beta^2 = \left(-S + V\frac{dS}{d\nu}\right)\Big/\gamma^2, \text{ \&c.},$$

$$\therefore \quad \left(\frac{dS}{d\lambda} - \frac{dS}{d\mu}\right) \Big/ (\alpha^2 - \beta^2) = \left(\frac{dS}{d\lambda} - \frac{dS}{d\nu}\right) \Big/ (\alpha^2 - \gamma^2) = \&c. \ldots\ldots (1).$$

This is the equation to a diameter of the circumspheric, and the point lies where this meets the spheric $S - \frac{V}{\alpha^2 - \beta^2}\left(\alpha^2 \frac{dS}{d\mu} - \beta^2 \frac{dS}{d\lambda}\right) = 0$,

or there are two such points on the diameter (1).

[More symmetrically, at the intersection of (1) and

$$\frac{n(n+1)}{2} S + V\left\{\frac{dS}{d\lambda}\left(\frac{\beta^2}{\alpha^2-\beta^2} + \frac{\gamma^2}{\alpha^2-\gamma^2} + \ldots\right) + \frac{dS}{d\mu}\left(\frac{\alpha^2}{\beta^2-\alpha^2} + \frac{\gamma^2}{\beta^2-\gamma^2} + \ldots\right) + \frac{dS}{d\nu}\left(\frac{\alpha^2}{\gamma^2-\alpha^2} + \frac{\beta^2}{\gamma^2-\beta^2} + \ldots\right) + \&c.\right\} = 0,$$

we have $$S - \left\{\lambda' \frac{dS}{d\lambda} + \mu' \frac{dS}{d\mu} + \nu' \frac{dS}{d\nu} + \ldots\right\} = 0,$$

a spheric, with centre $(\lambda', \mu', \nu' \ldots)$ and cutting $S = 0$ orthogonally, where

$$\lambda' = -\frac{2V}{n(n+1)}\left(\frac{\beta^2}{\alpha^2-\beta^2} + \frac{\gamma^2}{\alpha^2-\gamma^2} + \ldots\right),$$

$$\mu' = -\frac{2V}{n(n+1)}\left(\frac{\alpha^2}{\beta^2-\alpha^2} + \frac{\gamma^2}{\beta^2-\gamma^2} + \&c.\right), \quad \&c.,$$

and $$\alpha^2\lambda^2 + \beta^2\mu^2 + \gamma^2\nu^2 \ldots = 0.$$

Thus the perpendiculars upon the locus $\alpha^2\lambda + \beta^2\mu + \ldots = 0$ from the two points are equal and of opposite sign, and it is hence easily shown that each lies upon the linear polar of the other with respect to the circumspheric, and therefore the diameter (1) of these is divided harmonically.]

10435. (Maurice d'Ocagne.)—Soient F le point où la symédiane issue de A du triangle ABC coupe le cercle circonscrit à ce triangle, $F\beta$ et $F\gamma$ les perpendiculaires abaissées de ce point sur AB et sur AC. Si par les sommets B et C on mène deux antiparallèles quelconques BB′ et CC′ relativement à l'angle BAC, et que ces droites coupent respectivement $F\beta$ et $F\gamma$ en B_1 et en C_1, la parallèle à AB menée par B_1 et la parallèle à AC menée par C_1 se coupent sur la tangente en A du cercle ABC.

Solution by Professors Anderson, Nilkantha Sarkar, *and others.*

Since F is on the symmedian $c.F\gamma = b.F\beta$;

therefore, since $\angle FC\gamma = \angle ABF$, $c.C\gamma = b.B\beta$.

But CC′ and BB′ are antiparallels, so that $\angle \gamma CC' = \angle \beta BB'$. Hence $c.C'\gamma = b.B'\beta$; and this is plainly the condition that parallels through B′ and C′ to the sides BA and CA should meet on the tangent at A.

10486. (Professor DE LONGCHAMPS.) — On considère deux cercles Δ, Δ′, dont les centres sont les points O, O′. La droite OO′ rencontre Δ, Δ′, respectivement, aux points A, B; A′, B′. Soient M, M′ les extrémités de deux rayons mobiles, parallèles, et de même direction. Cela posé, AM, B′M′ se rencontrent en P; BM, A′M′ en Q. Démontrer que PQ passe constamment par un point fixe.

Solution by D. T. GRIFFITHS; Rev. T. R. TERRY, M.A.; *and others.*

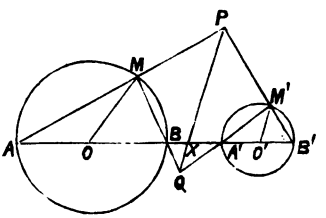
Let PQ meet OO′ at X; then the triangles BXQ, B′XP are similar; therefore BX : QX = B′X : PX.

The triangles QA′X, PAX are also similar; therefore

QX : A′X = PX : AX;

whence BX . AX = B′X . A′X.

Therefore X is a point on the radical axis of the two circles; that is to say, PQ passes through the point of intersection of the radical axis and the line joining centres.



8058. (W. J. C. SHARP, M.A.)—Prove that every unicursal twisted cubic curve is the partial intersection of three quadric surfaces, each pair of which has a common generator.

Solution by Professor SCHOUTE.

Let l_1, l_2, l_3 be three cords of the given twisted cubic R. Then (R, l_1), (R, l_2), (R, l_3) are the bases of three pencils of quadric surfaces. These pencils have, two by two, a surface in common; these three surfaces fulfil the conditions of the problem. When P_1, P_2, P_3 are three points of l_1, l_2, l_3, the three surfaces can be represented by the symbols (R, l_2, P_3), (R, l_3, P_1), (R, l_1, P_2), as well as by the symbols (R, l_2, l_3), (R, l_3, l_1), (R, l_1, l_2), &c.

If l_1, l_2, l_3 form a triangle, the surfaces are three cones, and the vertices of these cones the three vertices of the triangle.

10327. (E. M. LANGLEY, M.A.)—Deduce geometrically the existence of the symmedian point from that of the BROCARD-points.

Solution by JOHN J. BARNIVILLE.

Analysis.—Let the parallels through P, the symmedian point, meet the sides in DD′EE′FF′. Then ∠ DΩF = F′PE′, and DBF′ = DPF′; hence D, B, Ω, F are concyclic. Hence ΩOF = ΩBA, and since EDF = CBA,

ODE = OBC; but ODE = CDE, therefore OB and DE are parallel; in like manner OD and DF, OC and FE. Hence we have this

Construction.—Join ΩA, ΩB, ΩC, and inscribe in ABC a triangle DEF whose sides are parallel to ΩA, ΩB, ΩC. Through D, E, F draw parallels to the sides: these will intersect in the symmedian point.

9856. (J. Young, M.A. Extension of 9771.)—In a triangle the distance between the circumcentre and the orthocentre is equal to the difference of two sides $(a-b)$; prove that the angle C = 60°, or that

$$2\cos C + 3 = 4\cos(A-B).$$

Solution by Professors Beyens, A. Harkema, *and others.*

Si l'on appelle (d) la distance entre le centre et l'orthocentre du triangle, on a $d^2 = 9R^2-(a^2+b^2+c^2)$; mais

$$d = a-b = 2R(\sin A-\sin B);$$

donc $4(\sin^2 A+\sin^2 B-2\sin A\sin B) = 9-4(\sin^2 A+\sin^2 B+\sin^2 C)$,

et, ayant en égard que

$$A+B+C = 180° \quad \text{et} \quad \cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C = 1,$$

nous aurons après de réductions très faciles

$$12\cos^2 C-16\sin A\sin B\cos C+8\sin A\sin B-3 = 0,$$

d'où $\cos C = \frac{2}{12}\sin A\sin B \pm \frac{1}{12}(8\sin A\sin B-6)$,

d'où $\cos C = \frac{1}{2}$, $C = 60°$, $\cos C = \frac{1}{6}(8\sin A\sin B-3)$,

ou $6\cos C+3 = 8\sin A\sin B$, $2\cos C+3 = -4\cos C+8\sin A\sin B$;

mais $\cos C = -\cos A\cos B+\sin A\sin B$,

donc $2\cos C+3 = 4\cos(A-B)$.

10257. (Professor Sylvester, F.R.S.)—If $A_1, A_2 \ldots A_i)$ means the least of the quantities $A_1, A_2 \ldots A_i$, find (1) an expression for the rth in order of magnitude of the quantities $A_1, A_2 \ldots A_n$ in terms of the symbols ΣA_1; $\Sigma(A_1, A_2)$; $\Sigma(A_1, A_2, A_3)\ldots$; $\Sigma(A_1, A_2 \ldots A_{n-1})$; $(A_1, A_2 \ldots A_n)$, and show (2) the application of the question to the theory of life contingencies.

Solution by A. J. Pressland, M.A.

1. Suppose the quantities, rearranged in descending order of magnitude, to be $B_1, B_2 \ldots B_n$, we then get

$$-\Sigma(A_1)+B_1+B_2+B_3 \ldots\ldots +B_r+B_{r+1}\ldots\ldots +B_n = 0,$$
$$-\Sigma(A_1A_2)\ldots+B_2+2B_3 \ldots +(r-1)B_r+rB_{r+1}\ldots\ldots +(n-1)B_n = 0,$$
$$-\Sigma(A_1A_2\ldots A_r)\ldots\ldots\ldots\ldots B_r+rB_{r+1}\ldots+\frac{(n-1)!}{r!\,(n-r-1)!}B_n = 0,$$
$$-\Sigma(A_1\ldots A_n)\ldots\ldots\ldots\ldots +B_n = 0,$$

therefore

$$\begin{vmatrix} \Sigma(A_1), & 1, 1, 1, \ldots & 1, & 1, & 1, \ldots & 1 \\ \Sigma(A_1A_2), & 0, 1, 2, \ldots & r-2, & r, & \ldots & n-1 \\ \Sigma(A_1A_2A_3), & 0, 0, 1, \ldots & \frac{(r-2)(r-3)}{1.2}, & \frac{r(r-1)}{1.2} & \ldots & \frac{(n-1)(n-2)}{1.2} \\ \Sigma(A_1 \ldots A_r), & 0, 0, \ldots & 0, & r, & \ldots & \frac{(n-1)!}{r!(n-r-1)!} \\ \Sigma(A_1 \ldots A_n), & 0, 0, \ldots & \ldots & \ldots & \ldots & 1 \end{vmatrix}$$

$$= (-1)^{r+1} B \begin{vmatrix} 1, 1 & \ldots & 1 \\ 0, 1, 2 \ldots r, r+1 & \ldots & (n-1) \\ \ldots & \ldots & \ldots \\ 0 & \ldots 1, r \ldots & \frac{(n-1)!}{r!(n-r-1)!} \\ 0, 0 & \ldots & 1 \end{vmatrix}$$

[No solution has been sent to Part (2) of the Question.]

3737. (J. B. Sanders.)—To find the time in which a paraboloid of revolution whose altitude is h and parameter p, full of fluid, will empty itself through a small orifice at its vertex, its axis being vertical.

Solution by Professor Zerr; D. Biddle; *and others.*

For a vessel which is a surface of revolution the equation for the time of emptying itself is $t = -\pi \int \frac{y^2 dx}{k(2gx)^{\frac{1}{2}}}$,

where k is the area of the orifice. In this example $yz = px$; hence

$$t = -\frac{\pi p}{k(2g)^{\frac{1}{2}}} \int_0^h x^{\frac{1}{2}} dx = \frac{2}{3} \cdot \frac{\pi p h^{\frac{3}{2}}}{k(2g)^{\frac{1}{2}}}.$$

If r = radius of base, then $r^2 = ph$, and $t = \frac{2\pi r^2}{3k} \left(\frac{h}{2g}\right)^{\frac{1}{2}}$.

9331. (A. R. Johnson, M.A.)—Present (1) as the sums of finite series,

$$\sum_{r=1}^{r=\infty} 1/\{1+(2r+1)^{2p}\}, \quad \sum_{r=1}^{r=\infty} 1/\{1+(2r)^{2p}\},$$

p being any positive integer; and (2) prove that

$$\sum_{r=1}^{r=\infty} 1/(1+r^2+r^4) = -\tfrac{1}{2} + \frac{\pi}{2\sqrt{3}} \tanh(\tfrac{1}{2}\pi\sqrt{3}).$$

Solution by Professor SEBASTIAN SIRCOM, M.A.

(i.) Let the series be S and S_1; then, if

$$s_r = \frac{1}{3^r} + \frac{1}{5^r} + \ldots, \quad S = \frac{1}{2} + s_{2p} - s_{4p} + \ldots,$$

and the sum is obtained by selecting terms in the expansion in powers of x of $\frac{\pi x}{4} \tanh \frac{\pi x}{2} = \frac{x^2}{x^2+1} + \frac{x^2}{x^2+3^2} + \frac{x^2}{x^2+5^2} + \ldots = \frac{x^2}{x^2+1} + s_2 x^2 - s_4 x^4 + \ldots,$

and then putting $x = 1$.

Let, therefore, $\omega_n = \cos a_n + i \sin a_n$ be a root of ( )$^{1/2p}$, put $\omega_n x$ for x, and add all the results; then $\Sigma \left\{ \omega_n^2 x^2 / (\omega_n^2 x^2 + 1) \right\} = p$: and, putting $x = 1$,

$$\tfrac{1}{4}\pi \Sigma \left\{ \omega_n \tanh (\tfrac{1}{2}\omega_n \pi) \right\} = p + (-1)^p 2p (S - \tfrac{1}{2}).$$

Taking two conjugate roots together, $\cos a_n \pm i \sin a_n$, we have

$$\Sigma \left\{ \omega_n \tanh (\tfrac{1}{2}\omega_n \pi) \right\} = \sum_{n=1}^{n=p} \left(\frac{\cos a_n \sinh (\pi \cos a_n) - \sin a_n \sin (\pi \sin a_n)}{\cosh (\pi \cos a_n) + \cos (\pi \sin a_n)} \right),$$

and the sum is presented as required.

Similarly, if now $s_r = \frac{1}{2^r} + \frac{1}{4^r} + \frac{1}{6^r} + \ldots,$

since $\frac{\pi x}{4} \coth \frac{\pi x}{2} - \frac{1}{2} = \frac{x^2}{2^2+x^2} + \frac{x^2}{4^2+x^2} + \frac{x^2}{6^2+x^2} + \ldots = s_2 x^2 - s_4 x^4 + s_6 x^6 - \ldots,$

we have $\quad \tfrac{1}{4}\pi \Sigma \left\{ \omega_n \coth (\tfrac{1}{2}\omega_n \pi) \right\} - p = (-1)^p 2p S_1,$

and $\Sigma \left\{ \omega_n \coth (\tfrac{1}{2}\omega_n \pi) \right\} = \sum_{n=1}^{n=p} \left(\frac{\cos a_n \sinh (\pi \cos a_n) + \sin a_n \sin (\pi \sin a_n)}{\cosh (\pi \cos a_n) - \cos (\pi \sin a_n)} \right).$

(ii.) Since $\frac{1}{r^4 + r^2 + 1} = \frac{2r}{(2r+1)^2+3} - \frac{2(r-1)}{(2r-1)^2+3} + \frac{2}{(2r+1)^2+3};$

treating every term in the third series S′ in this way, except the first, and adding the results, we have

$$\frac{S'}{2} = \frac{1}{3^2+3} + \frac{1}{5^2+3} + \ldots,$$

therefore $\quad \frac{3S'}{2} = \frac{3}{3^2+3} + \frac{3}{5^2+3} + \ldots = \frac{\pi}{4}\sqrt{3} \tanh \frac{\pi\sqrt{3}}{2} - \frac{3}{1^2+3},$

$$S' = \frac{\pi}{2\sqrt{3}} \tanh \frac{\pi\sqrt{3}}{2} - \frac{1}{2}.$$

10468. (MAURICE D'OCAGNE.) — Soit μ la droite harmoniquement associée au point M, par rapport au triangle ABC. Les droites joignant un point quelconque de μ à A, à B, et à C, coupent les pédales correspondantes du point M en α, β, et γ. Démontrer que les côtés du triangle $\alpha\beta\gamma$ passent par les sommets du triangle ABC.

[La droite harmoniquement associée au point M par rapport au triangle ABC est celle qui joint les conjugués harmoniques, par rapport aux

extrémités des côtés de ce triangle, des points A′, B′, C′, où ces côtés sont respectivement coupés par AM, BM, CM. Les pédales du point M sont les droites A′B′, B′C′, C′A′.]

Solution by R. F. Davis, M.A.; Prof. Anderson, M.A.; *and others.*

Let the line μ intersect the sides of ABC in A″, B″, C″, so that $\{BA'CA''\}$, ... are harmonic. Then it can be easily shown that B′C′, C′A′, A′B′ pass respectively through A″, B″, C″. Let P be any point on the line μ; and let PB intersect AC, A′C′ in b, β, and PC intersect AB, A′B in c, γ, respectively. Then $\{Pb\beta B\}$, $\{PC\gamma c\}$ are each harmonic, and therefore $\beta\gamma$ passes through A.

[*Otherwise*:—Projecting μ to infinity so that M becomes the centroid of ABC, and A′, B′, C′ the mid-points of the sides, the mid-points β, γ of the two parallels Bb, Cc are obviously collinear with A.]

4054. (Rev. T. Mitchesson, B.A.)—In a spherical triangle, if t = the product of the tangents of the radii of the escribed circles, show that

$$\tan r = (n^4 \sin^{-2} s \,.\, t^{-1})^{\frac{1}{2}}.$$

Solution by W. J. Greenstreet, M.A.

$$(n^4 \,.\, \sin^{-2} s \,.\, t^{-1})^{\frac{1}{2}} = \left[\frac{n^4 \,.\, \sin(s-a)\sin(s-b)\sin(s-c)}{n^3 \,.\, \sin^2 s}\right]^{\frac{1}{2}}$$

$$\left(\because \tan r_a = \frac{n}{\sin(s-a)}, \text{ \&c.}\right)$$

$$= \left(\frac{n^6}{n^3 \sin^2 s}\right)^{\frac{1}{2}} = \frac{n}{\sin s} = \tan r.$$

9817. (W. J. C. Sharp, M.A.)—If A′, B′, C′ be the reflections of any point P, on the circumcircle of the triangle ABC, with respect to the sides; show, by Euclid, that A′, B′, C′ lie in a straight line which passes through the orthocentre of ABC. Hence deduce the theorem (Quest. 2145) that "the feet of the perpendiculars let fall on the sides of a triangle from any point on the circumscribing circle lie in a straight line. Show that this straight line is equidistant from the point and from the centre of perpendiculars of the triangle."

Solution by W. J. Greenstreet, M.A.; Professor Lampe; *and others.*

If A′B′C′ be any straight line, and P any point, and circles PA′B′, &c. be drawn; we can prove, by elementary Euclid, that the centres A, B, C

of these circles, and the point P, are concyclic; and also that PA′, &c. are common chords of the circles; therefore A′, B′, C′ are the reflections of P with respect to the sides. It is readily seen that, if U, V, W are the feet of perpendiculars from P on the sides AB, BC, CA of the triangle ABC, the segments UV, &c. are parallel to and equal to one half of the segments C′A′, &c.; therefore U, V, W are collinear, and UVW is equidistant from P and A′B′C′.

[The solution does not prove that the line A′B′C′ passes through the orthocentre; but if this point be denoted by K, the circles BKC, CKA, and AKB will pass through A′, B′, C′, respectively; and assuming that P lies on the arc AC of the circumcircle, and that PA′ and PB′ lie on the same side of PK as PC and PC′ on the others, we have

$$A'KC = A'BC = PBC;\ B'KC = PAC = PBC = A'KC;\ C'KA = C'BA = PBA;$$
$$B'KA = 180° - B'CA = 180° - PCA = 180° - PBA = 180° - C'KA;$$
$$A'KB = A'CB = PCB;\ C'KB = C'AB = PAB = 180° - PCB = 180° - A'KB;$$

therefore A′B′KC′ is a straight line. This proves the first part of the question, and Quest. 2145 at once follows.]

10332. (R. Knowles, B.A.)—Prove that (1) the trilinear coordinates of the centroid of the pedal triangle are proportional to

$$a^2 \cos(B-C),\quad b^2 \cos(A-C),\quad c^2 \cos(A-B);$$

and thence (2) that $a^3 \cos(B-C) + b^3 \cos(A-C) + c^3 \cos(A-B)$

$$= a(b^2+c^2)\cos A + b(a^2+c^2)\cos B + c(a^2+b^2)\cos C.$$

Solution by Rev. T. R. Terry, M.A.; G. G. Storr, M.A.; *and others.*

Let D, E, F be the feet of the perpendiculars, and α, β, γ the trilinear coordinates of the centroid of the triangle DEF. Then the perpendiculars on BC from D, E, F, are 0, $a \sin B \cos B$, $a \sin C \cos C$; therefore

$$6R\alpha = a^2 \cos(B-C) \quad \text{or} \quad ab \cos B + ac \cos C,$$

whence (1); also, each of the expressions (2) $= 6R(a\alpha + b\beta + c\gamma) = 12R\Delta$.

10209. (H. J. Woodall. Arising out of Quest. 10048.)—If a voter may use r, $r-1$, $r-2$, ... 2, 1, 0 of his votes, show that the number of ways in which he may now act is $(n+r)! / (n! \times r!)$.

Solution by the Proposer.

Let there be n candidates, then the effect of the question is that there shall be an extra receptacle for the unused votes, in effect $n+1$ candidates. Therefore the stated result follows at once.

10489. (Professor Déprez.)—On donne dans un plan deux triangles ABC, A′B′C′ ; trouver le lieu d'un point P′ tel que les perpendiculaires abaissées de A sur A′P′, de B sur B′P′, de C sur C′P′ concourent en un même point P. Trouver également le lieu du point P.

10425. (Professor Tarry.)—Quatre trains se meuvent sur des voies rectilignes avec des vitesses uniformes. On donne leurs positions à deux instants différents. Construire une voie rectiligne qui puisse être parcourue par un train avec un vitesse uniforme telle que les quatre premiers trains paraissent immobiles au conducteur du cinquième.

Solution by Professor Schoute; A. J. Pressland, B.A.; *and others.*

(10489.) Nous cherchons d'abord la relation entre les points P_a et P'_a (Fig. 1), choisis de manière que les angles $(BP_a, B'P'_a)$ et $(CP_a, C'P'_a)$ sont droits. Cette relation est une transformation quadratique non involutive. Les points fondamentaux du système plan P_a sont B, C et le point à l'infini commun aux normales à B′C′ ; les points fondamentaux du système plan P'_a sont B′, C′ et le point à l'infini commun aux normales à BC. Les points de la droite l_∞ à l'infini se correspondent à eux-mêmes.

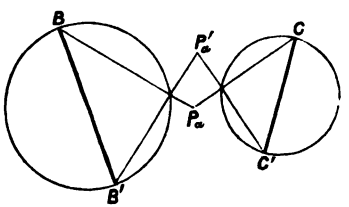


Fig. 1.

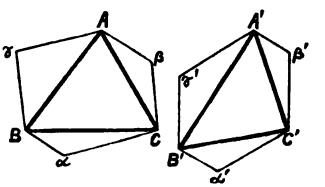


Fig. 2.

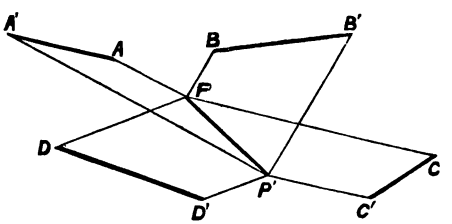


Fig. 3.

La relation entre les points P_b et P'_b, choisis de manière que les angles $(CP_b, C'P'_b)$ et $(AP_b, A'P'_b)$ sont droits, est précisément de la même nature. Et chaque point $P_a = P_b$, pour lequel les points correspondants P'_a et P'_b coincident est un point du lieu décrit par P. Cherchons combien de points P se trouvent sur une droite quelconque l. Si $P_a = P_b$ parcourt la droite

l, les points correspondants décrivent deux hyperboles qui se coupent en quatre points, le point C, le point de l à l'infini et deux autres points à distance finie. Donc le lieu de P se compose de la droite l_∞ et d'une conique ; de la même manière le lieu de P′ se compose de la droite l_∞ et d'une autre conique.

De chacune de ces deux coniques on trouve immédiatement six points. Ainsi, la première passe par A, B, C et les intersections α, β, γ des perpendiculaires par A, B, C sur les côtés de l'autre triangle, &c. Et ces deux sextuples de points se trouvent sur deux coniques, eu égard le théorème de Pascal.

Si les deux triangles ABC et A′B′C′ admettent le même cercle circonscrit, la transformation est involutive, et les deux lieux coincident avec le cercle circonscrit commun.

Le problème peut être étendu à deux quadrangles dans la forme suivante :

"On donne dans un plan deux quadrangles ABCD et A′B′C′D′ ; trouver un point P tel que les perpendiculaires de A′ sur AP, de B′ sur BP, de C′ sur CP, et de D′ sur DP concourent en un même point P′ (deux solutions.)"

Evidemment on trouve les points P en considérant d'abord les triangles ABC et A′B′C′ et ensuite les triangles BCD et B′C′D′. Des deux lieux géométriques correspondants de P, le premier est une conique par A, B, C, et le second une conique par B, C, D. Ces deux coniques par B et C admettent deux autres points communs, les deux points P qui se rapportent aux deux solutions.

(10425.) Ce problème peut se mettre dans la forme géométrique suivante. Dans un plan on donne quatre segments de droite AA′, BB′, CC′, DD′ (Fig. 3) ; chercher un segment PP′, qui soit à la fois une projection inclinée des quatre segments donnés. Dans cette forme ABCD et A′B′C′D′ désignent deux quadruples de positions simultanées des quatre trains. On a donc à résoudre le problème suivant :

"On donne dans un plan deux quadruples ABCD et A′B′C′D′ ; trouver un point P tel que les parallèles à AP par A′, à BP par B′, à CP par C′, et à DP par D′ passent par un même point P′."

Dans cette forme la solution de ce problème peut être copiée de celle du précédent en remplaçant l'angle droit par l'angle zéro. Ainsi, quand on ne considère que les segments BB′, CC′, on trouve que la relation entre les points P_a et P'_a est une transformation quadratique non involutive, que les points fondamentaux du système plan P_a sont B, C et le point à l'infini communs aux parallèles à B′C′, &c.

Eu egard à l'extension du problème précédent, on trouve donc deux solutions.

L'extension de ce problème-ci à l'espace prend la forme suivante tout simple :

"Trois ballons sont animés d'un mouvement uniforme rectiligne. Déterminer le mouvement d'un quatrième ballon d'où les trois ballons paraissent immobiles."

Soient A, B, C et A′, B′, C′ deux triples de positions simultanées des trois ballons donnés. On ne trouve qu'un seul point P le point d'intersection des trois plans par BC, CA, AB parallèles respectivement à B′C′, C′A′, A′B′, &c.

1423, 2316, 3941, 4794. (By the late Prof. CLIFFORD, F.R.S.)—Prove that $\int_0^{\frac{1}{2}\pi} \cos(a\tan x)\,.\,\epsilon^{\beta\tan x}\,\delta x = \frac{1}{2}\pi\epsilon^{-\alpha}(\cos\beta+\sin\beta)$.

Solution by the Rev. H. LONDON, M.A.

$$\int_0^{\frac{1}{2}\pi} \cos(a\tan x)\,\epsilon^{\beta\tan x}\,\delta x = \int_0^{\infty} \frac{\cos az\,.\,\epsilon^{\beta z}\,\delta z}{1+z^2} \text{ (where } x=\tan^{-1}z)$$

$$= \int_0^{\infty} \frac{\cos az}{1+z^2}\left(1+\beta z+\frac{\beta^2 z^2}{2!}+\frac{\beta^3 z^3}{3!}\ldots\right)\delta z.$$

Now $\int_0^{\infty} \frac{\cos az}{1+z^2}\,\delta z = \frac{1}{2}\pi\epsilon^{-a}$. Differentiating with regard to a,

$$\int_0^{\infty} \frac{-z\sin az}{1+z^2}\,\delta z = -\tfrac{1}{2}\pi\epsilon^{-a};\quad \therefore \int_0^{\infty} \frac{z\sin az}{1+z^2}\,\delta z = \tfrac{1}{2}\pi\epsilon^{-a},$$

$$\int_0^{\infty} \frac{z^2\cos az}{1+z^2}\,\delta z = -\tfrac{1}{2}\pi\epsilon^{-a},\quad \int_0^{\infty} \frac{z^3\sin az}{1+z^2}\,\delta z = -\tfrac{1}{2}\pi\epsilon^{-a},\ \&c.$$

Assume $$\int_0^{\infty} \frac{z^r\sin az}{1+z^2}\,\delta z = \int_0^{\infty} \frac{z^r\cos az}{1+z^2}\,\delta z;$$

therefore $$\int_0^{\infty} \frac{\cos az}{1+z^2}\left(1+\beta z+\frac{\beta^2 z^2}{2!}+\frac{\beta^3 z^3}{3!}+\ldots\right)\delta z$$

$$= \tfrac{1}{2}\pi\epsilon^{-a}\left(1+\beta-\frac{\beta^2}{2!}-\frac{\beta^3}{3!}+\frac{\beta^4}{4!}+\frac{\beta^5}{5!}-\ldots\right) = \tfrac{1}{2}\pi\epsilon^{-a}(\cos\beta+\sin\beta).$$

[The Solver is in doubt as to how far his assumption is justifiable. The mean values of $\sin az$ and $\cos az$ between the given limits being equal, there is some reason for assuming their integrals to be so, but Mr. LONDON states that he has not been able to obtain any direct proof.]

10534. (Rev. H. SEWELL, B.A.)—If $a, b, c, \ldots k$ be n positive quantities, prove that $\left(\frac{a+b+c+\ldots+k}{n}\right)^{a+b+c+\ldots+k} < a^a b^b c^c \ldots k^k$.

Solution by D. BIDDLE; *the* PROPOSER; *and others.*

Beginning with two quantities only, let $a<k$; then

$$\left(\frac{a+k}{2}\right)^{a+k} : a^a k^k = (a+k)^a\,.\,(a+k)^k : (2a)^a\,.\,(2k)^k$$

$$= \left(\frac{a+k}{2a}\right)^a : \left(\frac{2k}{a+k}\right)^k = \left(1+\frac{k-a}{2a}\right)^a : \left(1+\frac{k-a}{a+k}\right)^k;$$

and, since $\frac{a}{2a} < \frac{k}{a+k}$, whilst $\frac{1}{2a} > \frac{1}{a+k}$, it is easy to find, by expansion,

that $\left(1+\frac{k-a}{2a}\right)^a < \left(1+\frac{k-a}{a+k}\right)^k$; therefore $\left(\frac{a+k}{2}\right)^{a+k} < a^a k^k$.

Where the quantities are more numerous, the same law is proved by taking a series of means, whence

$$\left(\frac{a+k}{2}\right)^{a+k} \cdot \left(\frac{b+i}{2}\right)^{b+i} \ldots < a^a b^b \ldots i^i k^k.$$

This suffices when the quantities are consecutive integers, wherever beginning, or when, integral or not, they are balanced in couples about an identical mean. In other cases, the means of previous means must be taken, whence

$$(\mu_1^{m_1} \cdot \mu_1^{m_1})(\mu_2^{m_2} \cdot \mu_2^{m_2}) \ldots < (m_1^{m_1} \cdot m_1^{m_1})(m_1^{m_2} \cdot m_2^{m_2}) \ldots < a^a b^b \ldots i^i k^k.$$

Then those of each succeeding series will more closely approximate to each other, whilst their number (n), including repetitions, and their sum total will remain the same throughout. Consequently, when the final mean is reached, we have M^{nM}, which $= \left(\frac{a+b+\ldots+k}{n}\right)^{a+b+\ldots+k}$; and, by a sorites, we conclude that $M^{nM} < a^a b^b \ldots i^i k^k$.

10515. (Professor CATALAN.)—Si les diagonales AD, BE, CF d'un hexagone ABCDEF sont égales, et que les côtés soient parallèles, deux à deux, l'hexagone est inscriptible à une circonférence.

Solution by R. KNOWLES, B.A.; Prof. ANDERSON, M.A.; *and others.*

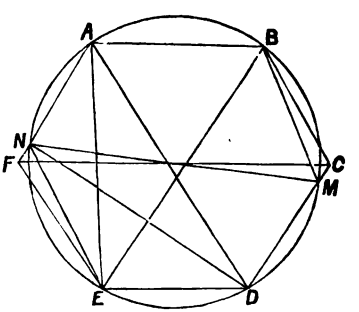
Since AB and ED are parallel, the lines AD and BE, being equal in length, are equally inclined to AB and ED; therefore

$$\angle ADE = BED.$$

The triangles ADE and BED have, therefore, two sides and the contained angle of one equal to two sides and the contained angle of the other. Hence

$$\angle EAD = EBD,$$

and the points A, B, D, E lie on a circle. Let this circle meet AF in N, and CD in M. Join MN, AE, ND.



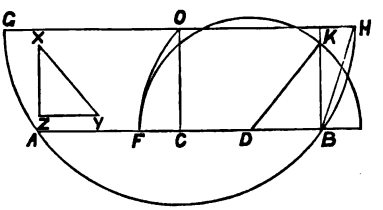
Then $\angle EDM = BAN$, $\angle NAE = NDE$; $\therefore \angle BAE = NDM$,

$\therefore BE = NM$, $\therefore NM = AD = FC$;

consequently NM coincides with FC or AD. But it cannot coincide with AD, for then AF and DC would be two parallel tangents to the circle, and AD a diameter, and FC could not be equal to AD. Hence it coincides with FC, which proves the proposition.

10539. (Hilda Hudson.)—Through two points (A, B) draw a circle so that the end (H) of the diameter (GH) parallel to AB may be at a given distance from the point B.

Solution by M. Brierley; Profs. Anderson *and* Hudson; *and others.*



Let XY be the given distance. At Y, X make the angles YXZ, XYZ each half right angle. Bisect AB in C, and CB in D; draw BK at right angles to AB, and equal to XZ. With centre D and radius DK draw a circle cutting AB in F; and with centre B and radius BF draw another circle cutting a perpendicular CO to AB in O. Then O will be the centre of the required circle. For we have

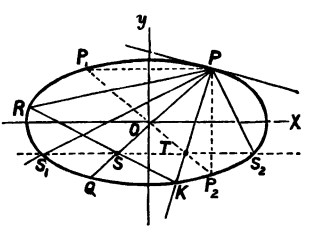
$BH = OF$, $OF^2 = 2FC \cdot FB = 2BK^2 = 2XZ^2 = XY^2$; $\therefore BH = XY$.

9294. (Belle Easton.)—PK is a chord of an ellipse normal at P, PQ, PR two chords such that the straight lines bisecting the angles between them are equally inclined to PK; prove that the locus of the point of intersection of PQ, KR is the straight line

$$\left\{(a^2+b^2)\,b^2-(a^2-b^2)^2\sin^2\phi\right\}\frac{x\cos\phi}{a}+\left\{(a^2+b^2)\,a^2-(a^2-b^2)^2\cos^2\phi\right\}\frac{y\sin\phi}{b}$$
$$=(a^2-b^2)\,(b^2\cos^2\phi-a^2\sin^2\phi),$$

where ϕ is the eccentric angle of the point P.

Solution by Professor Schoute.



1. The locus of the point S common to PQ and KR is a straight line. For the pencils of rays PQ and PR are in isogonal hyperbolical involution. Therefore the pencils of rays PQ and KR are homographical, *i.e.*, in (1, 1) correspondence, and, as Q is in K when R is in P, they generate an improper conic consisting of the straight line PK and another straight line, the latter of which forms the locus in question.

2. The locus passes through the points S_1 and S_2, where the given conic meets the lines through P, forming with PK angles $\frac{1}{4}\pi$. It also contains the point T of PK, where, according to a known theorem, the normal PK is

met by all the chords that are seen from P under a right angle and the pole of PK. We determine the equation of the locus by means of the point T and this pole.

3. When ϕ is the eccentric angle of P, and $2a$, $2b$ are the axes of the conic, the equations

$$ax \sin\phi - by\cos\phi = (a^2-b^2)\sin\phi\cos\phi, \quad bx\sin\phi + ay\cos\phi = 0$$

represent the normal PK and the diameter P_1P_2 passing through the symmetric points of P. So the equation of the locus can be thrown into the form

$$(a+kb)\,x\sin\phi - (b-ka)\cos\phi = (a^2-b^2)\sin\phi\cos\phi,$$

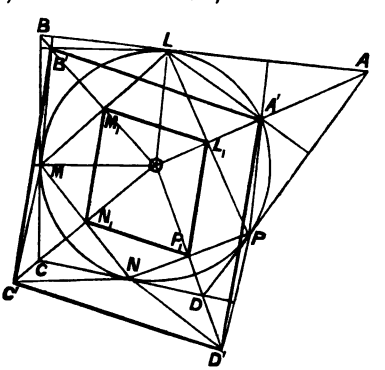
where the parameter k is to be found by the condition that the pole of this line is on PK. Now the pole of the line is the point with the coordinates $\quad a^2(a+kb)/(a^2-b^2)\cos\phi, \quad -b^2(b-ka)/(a^2-b^2)\sin\phi.$

So k is determined by the condition

$$kab\,(a^2\sin^2\phi - b^2\cos^2\phi) = (a^2-b^2)^2\sin^2\phi\cos^2\phi - a^4\sin^2\phi - b^4\cos^2\phi, \text{ \&c.}$$

10519. (Professor DÉCAMPS.)—Dans tout quadrilatère circonscriptible, les points de rencontre des hauteurs des quatre triangles formés par deux côtés consecutifs et leur corde de contact avec le cercle sont les sommets d'un parallélogramme.

Solution by H. J. WOODALL ; Professor ANDERSON ; *and others.*



Let ABCD be the given quadrilateral touching the in-circle at LMNP. Join centre O with A, B, C, D, and draw LM, MN, NP, PL; then therefore AO, BO, CO, DO bisect LP, LM, LN, NP at right angles at L_1, M_1, N_1, P_1, then L_1, M_1, N_1, P_1 is a parallelogram (because the sides are the joins of the mid-points of a quadrilateral).

Now OL, MB′ are perpendicular to AB, and likewise OM, LB′ to BC; hence LB′MO is a parallelogram; therefore $OM_1 = \frac{1}{2}OB'$,

$$OL_1 = \tfrac{1}{2}OA', \quad ON_1 = \tfrac{1}{2}OC', \quad OP_1 = \tfrac{1}{2}OD';$$

consequently A′B′, B′C′, C′D′, D′A′ are parallel to M_1N_1, N_1P_1, P_1L_1; therefore A′B′C′D′ is a parallelogram.

10538. (W. J. C. Sharp, M.A.)—A flexible hawser is fastened to a point at the bottom of a stream, in which the force of the current varies as any function of the depth. Find the differential equation to the catenary in which it hangs, and show that, if the axis of y be vertical, m be the weight of the unit of length of the hawser when weighed in water, b an arbitrary constant, and T the tension, $T = (ms - b)\,ds/dy$.

Solution by D. Edwardes, B.A.; Prof. Chakrivarti; *and others.*

Measuring the y-axis vertically upwards, the x-axis in the direction of the stream, and denoting by $\phi(y)$ the pressure on a vertical unit length of hawser, the equations of equilibrium are

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) + \phi(y)\frac{dy}{ds} = 0, \quad \frac{d}{ds}\left(T\frac{dy}{ds}\right) - m = 0.$$

Integrating the second equation, $T = (ms - b)\dfrac{ds}{dy}$, and eliminating T between this and the first equation,

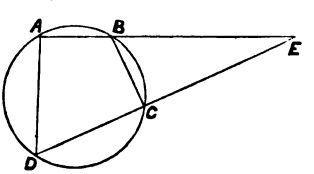
$$(ms - b)\frac{d^2x}{dy^2} + m\frac{dx}{dy}\frac{ds}{dy} + \phi(y) = 0,$$

$$\therefore \quad m\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}^{\frac{1}{2}} + \frac{d}{dy}\left[\phi(y) + m\frac{dx}{dy}\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}^{\frac{1}{2}} \Big/ \frac{d^2x}{dy^2}\right] = 0.$$

[This theorem is really the same as Quest. 7203, which is itself derived from Prof. Clifford's Kite-string Question 6009.]

10185. (Professor Wolstenholme, D.Sc.)—Prove by Statical principles that, if A, B, C, D be four points on a circle, the poles of AC, BD with respect to the circle lie on the same straight line with the intersections of AB, CD, and of AD, BC.

Solution by Professors Wolstenholme, Sarkar, *and others.*



Suppose four forces to act along AB, AD, CB, CD, whose magnitudes are inversely as the lengths of the chords along which they act, then the resultant of the forces along AB, AC will make with AB an angle equal to that which the bisector of BD through A makes with AD (since this last would be the direction of the resultant if the forces were as the lengths), or the resultant will act along the symmedian through A of the triangle ABD, and will therefore pass through the pole of BD. Similarly, the resultant of the forces along CB, CD will also pass through the pole of BC, and therefore the resultant of all four forces will pass through the pole of BC.

In exactly the same way, it may be proved that the resultant of the

forces along AB, CB, and that of the forces along AD, CD, will pass through the pole of AC; so that the resultant of all four forces passes through the pole of AC. If AB, CD intersect in E, and p_1, p_2 be the perpendiculars from E on BC, AD, then, since the triangle EBC, EDA are similar, $p_1 : BC = p_2 : DA$; hence the moments of the forces along CB, AD about E are equal and opposite, or the resultant of the four forces passes through E. So, if BC, AD intersect in F, the resultant passes through F; and therefore the four points, the poles of BD, AC, and the intersections of AB, CD, and of AD, BC, are collinear.

The equation of EF, referred to quadrilinear coordinates (the distances α, β, γ, δ from AB, BC, CD, DA) is $\alpha/a - \beta/b + \gamma/c - \delta/d = 0$, where a, b, c, d are the lengths AB, BC, CD, DA. Also, by taking forces along CA, CB; DA, DB inversely as the lengths of the chords along which they act, we can prove that the poles of AB, CD are collinear with FG, where G is the join of AC, BD; and, similarly, that the poles of BC, AD, G, E are collinear.

10530. (Professor EMMERICH.)—If ω be the Brocard-angle, and δ any acute angle of a triangle, prove that $\cot\omega \sin 2\delta > 2$ in an obtuse-angled triangle, and < 2 in an acute-angled triangle.

Solution by Professor ANDERSON, M.A.; R. KNOWLES, B.A.; *and others.*

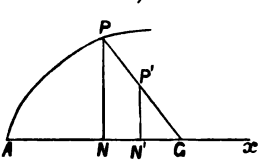
Let β, γ, δ be the angles of a triangle; then $\cot\omega = \cot\beta + \cot\gamma + \cot\delta$; therefore

$$\cot\omega \sin 2\delta = 2\cos\delta \{\sin(\beta+\gamma)(\cot\beta + \cot\gamma) + \cos\delta\}$$
$$= 2\cos^2(\beta+\gamma) + 2\sin^2(\beta+\gamma)\cos\delta/\sin\beta\sin\gamma.$$

This is $>$ or < 2, according as $\cos\delta/\sin\beta\sin\gamma$ is $>$ or < 1. But $\cos\delta/\sin\beta\sin\gamma = 1 - \cot\beta\cot\gamma$, which is < 1 if β and γ are both acute, and > 1 if one of them be obtuse.

10175. (Professor TRUC.)—Trouver le lieu géométrique des milieux normales à une parabole donnée en supposant ces normales limitées à l'axe.

Solution by Rev. T. GALLIERS, M.A.; Professor BEYENS; *and others.*



Let P′ be the middle point of PG, P′N′ the ordinate of P′. By properties of the parabola

$$PN^2 = 4AS \cdot AN, \quad NG = 2AS.$$

Hence $NN' = AS$;

also, $PN = 2P'N'$;

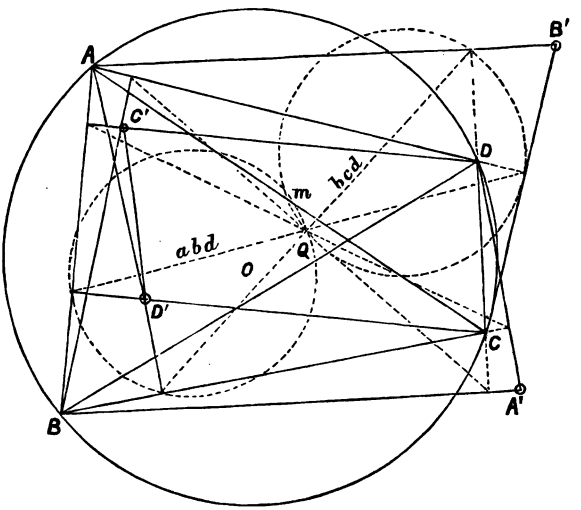
therefore $P'N'^2 = AS \cdot SN'$.

Or the locus of P is a parabola, having its vertex at S, axis Ax, and latus rectum $= AS$.

9997. (J. C. St. Clair.)—If A, B, C, D be four points on a circle, prove that (1) the four nine-points circles of the four triangles ABC, &c. pass through the same point Q; (2) the Simson-lines of each triangle with respect to the fourth point pass through Q; (3) the four orthocentres form a quadrilateral equal to ABCD, and in perspective with it at centre Q; and (4) if a fifth point be taken on the circle, the five quadrilateral Q points lie on a circle, and so on.

Solution by the Proposer.

(1) If at the points A, B, C, D tangents be drawn to the circle, their intersections are the inverses, with respect to the circle, of the middle points of the sides and diagonals of the quadrilateral ABCD. Hence the



circumcircles of the four triangles formed by the tangents are the inverses of the four nine-points circles of ABC, &c. (see Casey's *Sequel to Euclid*, Bk. vi., Sec. iv., Prop. 12). But, by a well-known theorem, the four circumcircles of the triangles formed by four lines pass through a common point (Casey's *Sequel to Euclid*, Bk. iii., 12, Cor. 3), and therefore the four nine-points circles of the triangles ABC, &c. have a common point Q.

(4) This follows from Clifford's extension of the theorem referred to.

(2) The extremities of the Simson-lines *bcd*, *abd* are the intersections of a circle upon AC as diameter with the two equal nine-points circles (*abc*), (*acd*) which intersect in Q and in *m* the middle point of AC. If *m* be joined to these extremities, it is easily shown that the lines pass through Q; and similarly for the other two lines *abc*, *acd*.

(3) Denoting by D′ the orthocentre of ABC, &c., we have

$$\angle BC'D = \text{supt. } BAD = BCD \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

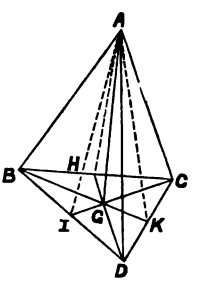
Also $\angle AC'B = AD'B$, and therefore $\angle BC'D = BAD' = BCD' \ldots\ldots (2)$.

Subtracting (2) from (1), we have $\angle DC'D = DCD'$. Hence, since DC′ is parallel to CD′, DCD′C′ is a parallelogram, and the diagonals DD′, CC′ bisect each other. But they are bisected respectively by the Simson-lines *abc*, *abd* (Casey's *Sequel*, III., 14); hence their point of intersection is Q.

In the same way, it may be shown that AA′, BB′ pass through, and are bisected at, Q.

10518. (Professor Schnbvikoff.)—Soient A le sommet d'un tétraèdre, BCD sa base, G le point d'intersection des médianes du triangle BCD. Démontrer que $3AB^2 + 3AC^2 + 3AD^2 = BC^2 + BD^2 + CD^2 + 9AG^2$.

Solution by Professors Anderson, Zerr, Aiyar, *and others.*



Since G is the centroid of the triangle BCD,

$$AB^2 + AC^2 + AD^2 = BG^2 + CG^2 + DG^2 + 3AG^2;$$

and, similarly, or directly from this equation by making A coincide with B, we have

$$BC^2 + BD^2 = 4BG^2 + CG^2 + DG^2;$$
$$BC^2 + CD^2 = BG^2 + 4CG^2 + DG^2,$$
$$BD^2 + CD^2 = BG^2 + CG^2 + 4DG^2;$$

hence, adding, we obtain the result.

10294. (M. Thénard.)—Construire un triangle isoscèle ABC, connaissant un des deux côtés égaux et la somme BC + CD de la base BC et de sa projection CD sur AC.

Solution by J. C. St. Clair; J. J. Barniville; *and others.*

Let $AC = a$ and $BC + CD = k$; then $(AB)^2 = (BC)^2 + a^2 - 2CD \cdot a$;

$\therefore (BC)^2 = 2a \cdot CD = 2a(k - BC)$; $\therefore (BC + a)^2 = a(a + 2k)$.

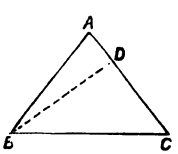


Fig. 1.

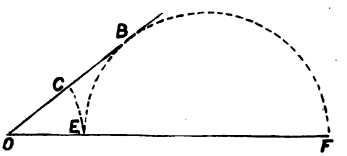


Fig. 2.

Take (Fig. 2) OE = a and produce it to F, making EF = $2k$. Upon EF describe a circle and draw the tangent OB. Take OC = OE. Then BC is the base of the required triangle; and the side a being known it is easily constructed.

[BC + CD = $a(1+\cos C) = 2a\cos^2 C = a + a^2/2c$; and if BC + CD = d, $a = -c \pm (c^2+2cd)^{\frac{1}{2}}$; hence the triangle is solved.]

10510. (D. T. Griffiths.)—The three tangents to a parabola at the points A, B, C form an equilateral triangle. Prove that the ellipse which touches sides of the triangle ABC at their mid-points passes through the focus of the parabola.

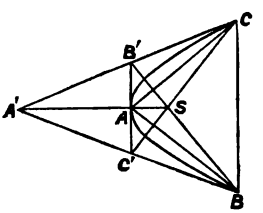
Solution by Professors Anderson, Chakrivarti, *and others.*

If the parabola be referred to the equilateral triangle as triangle of reference, its equation is

$$(lx)^{\frac{1}{2}} + (my)^{\frac{1}{2}} + (nz)^{\frac{1}{2}} = 0,$$

with the condition $l+m+n = 0$; and the focus is given by $xl = ym = zn$. The equation of the ellipse referred to the triangle ABC is $x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = 0$,

which, when transformed to the equilateral triangle, becomes



$$n(lx+my-nz)^{\frac{1}{2}} + l(-lx+my+nz)^{\frac{1}{2}} + m(lx-my+nz)^{\frac{1}{2}} = 0.$$

This equation is satisfied by the coordinates of the focus, since $l+m+n=0$.

10212. (W. J. C. Sharp, M.A.)—Solve (1) the equation of differences

$$(n+1)P_{n-1} - (2n+1)P_n + nP_{n-1} = 0;$$

find (2) what this solution becomes when $P_1 = P_2 = 1$; and hence (3) show that Legendre's coefficients are each = 1, when $x = 1$.

Solution by Rev. J. L. Kitchin, M.A.; E. Rutter; *and others.*

This may be written $(n+1)(P_{n+1}-P_n) = n(P_n - P_{n-1})$;

therefore $n(P_n - P_{n-1}) = (n-1)(P_{n-1} - P_{n-2})$, &c. = &c.;

$$3(P_3 - P_2) = 2(P_2 - P_1);$$

therefore $(n+1)(P_{n+1}-P_n) = 2(P_2 - P_1) = c$, a constant;

$\therefore$ $P_{n+1} - P_n = c/(n+1)$, $P_n - P_{n-1} = c/n$, &c. = &c., $P_2 - P_1 = c/2$;

therefore $P_{n+1} = P_1 + 2(P_2 - P_1)[\frac{1}{3} + \frac{1}{5} + \ldots + 1/(1+n)]$,

therefore $P_{n+1} = 1$, when $P_1 = P_2 = 1$.

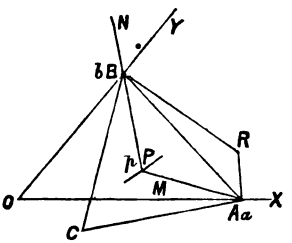
The equation $(n+1)P_{n+1} - (2n+1)xP_n + nP_{n-1} = 0$ is a relation between P_{n+1}, P_n, P_{n-1}, Legendre's coefficients. When $x = 1$, this becomes the equation in question, and P_{n+1}, P_n, ... all $= 1$.

10223. (Professor NEUBERG.)—Une droite AB de longueur constante se meut entre deux droites fixes OX, OY. On joint ses extrémités à un point fixe C du plan XOY. Déterminer géométriquement les points de contact de hauteurs AA', BB' du triangle ABC avec leurs enveloppes, ainsi que la tangente à la trajectoire de l'orthocentre.

Solution by Professor RAMASWAMI AIYAR.

Let ab be a position of the line infinitely near to AB; P, p the orthocentres of triangles CAB, Cab; M the intersection of AP, ap; and N that of BP, bp; then

$$\angle AMa = BCb;$$



therefore the radii of circles AMa, BCb are to one another as Aa : Bb, that is, as RA : RB, if these be perpendiculars at A, B to OX, OY. Thus the circle AMa is determined, and therefore the point M Similarly we may find N. Again, since the angles PMp, PNp = angles AMa, ACa, we see that the line Pp is such that the circles touching it at P and passing through the known points M, N are to one another as the known circles AMa, ACa.

9213. (D. BIDDLE.)—A floor is traversed by parallel equidistant lines, and a triangular plate, the diameter of whose circumcircle is equal to the distance between the lines, is thrown down at random. Given, at the circumcentre, the angles α, β, γ, which the sides of the plate subtend, find the probability that the plate lies wholly between two lines.

Solution by the PROPOSER.

By reference to the solution of 10219 (Vol. LII., p. 97), it is easy to see that the probability of the triangle being *intersected* by a parallel is given by the perimeter of the triangle divided by the perimeter of the circumcircle, since the latter just touches two adjacent parallels when placed between them. The probability in the question is the complement of this, therefore

$$P = 1 - (\sin\tfrac{1}{2}\alpha + \sin\tfrac{1}{2}\beta + \sin\tfrac{1}{2}\gamma)/\pi.$$

10509. (R. H. W. WHAPHAM, B.A.)—In an epicycloid, let C be the centre of the fixed circle, C′ that of the rolling circle; P any point on the epicycloid; O the corresponding point of contact of the two circles. Prove the following construction for the centre of curvature at P:—Produce CC′ to K, such that C′K = CC′. Join KP. Through C draw CT parallel to KP, meeting PO in T; then T is the centre of curvature at P.

Solution by D. T. GRIFFITHS; Professor ZERR; *and others.*

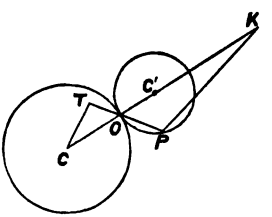
Triangles COT, KOP are evidently similar; therefore

$$CO : OT = KO : OP;$$

therefore $$OT = \frac{CO \cdot OP}{KO},$$

i.e., $$OT = \frac{CO}{CO + 2C'O} \cdot OP.$$

Whence, by Art. 287, WILLIAMSON'S *Diff. Calc.*, T is the centre of curvature corresponding to P.



10168. (H. W. SEGAR.)—A side of a square moves with its ends on two straight lines at right angles, and touches a circle which touches those straight lines. Show that the other corners lie on two hyperbolas; find their equations referred to the fixed lines as axes.

Solution by Rev. J. L. KITCHIN; R. KNOWLES, B.A.; *and others.*

Let r = radius of circle,

a = side of square,

OA = b, OD = c;

then BM = b, AM = c;

$\therefore\ x = b + c,\quad y = b;$

$\therefore\ c = x - y,\ \therefore\ a^2 = (x-y)^2 + y^2.$

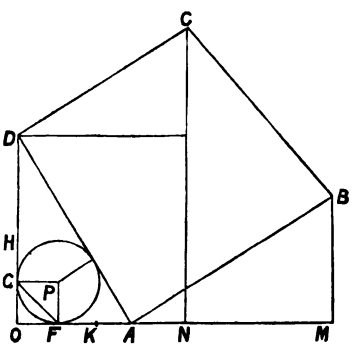
Also, $a + 2r = b + c = x;$

therefore $a = x - 2r,$

$\therefore\ (x-y)^2 + y^2 = (x-2r)^2,$

or $y^2 - xy + 2rx = 2r^2,$

which designates an hyperbola.



For C, $x = c,\ y = b + c;$

therefore $b = y - x,$

therefore $a^2 = x^2 + (y-x)^2 = (y-2r)^2,\quad \therefore\ x^2 - xy + 2ry = 2r^2,$

the locus of C.

Subtracting one from the other, we get $(x-y)(x+y-2r) = 0$; or $x - y = 0,\ x + y - 2r = 0$ are common chords of the two conics.

2322. (The late Professor Townsend, F.R.S.)—Express the radius (R) of a circle orthogonal to three others in a plane, in terms of their three radii (p, q, r) and the three sides (a, b, c) of the triangle determined by their three centres. Investigate the corresponding formula for the radius of the circle orthogonal to three others on the surface of a sphere.

Solution by W. J. Curran Sharp, M.A.

The well-known relation between the mutual distances of four points in a plane may be put in the form (*Proc. Lond. Math. Soc.*, xviii., p. 321)

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 1, & 0, & (1.2)^2, & (1.3)^2, & (1.4)^2 \\ 1, & (2.1)^2, & 0, & (2.3)^2, & (2.4)^2 \\ 1, & (3.1)^2, & (3.2)^2, & 0, & (3.4)^2 \\ 1, & (4.1)^2, & (4.2)^2, & (4.3)^2, & 0 \end{vmatrix} = 0,$$

where (1.2), &c. stand for the distances. In the question

$$(1.2)^2 = R^2 + p^2, \quad (1.3)^2 = R^2 + q^2, \quad (1.4)^2 = R^2 + r^2,$$
$$(2.3)^2 = c^2, \quad (2.4)^2 = b^2, \quad (3.4)^2 = a^2,$$

and

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1, \\ 1, & 0, & R^2+p^2, & R^2+q^2, & R^2+r^2 \\ 1, & R^2+p^2, & 0, & c^2, & b^2 \\ 1, & R^2+q^2, & c^2, & 0, & a^2 \\ 1, & R^2+r^2, & b^2, & a^2, & 0 \end{vmatrix} = 0,$$

or

$$2R^2 \begin{vmatrix} 0, & 1, & 1, & 1 \\ 1, & 0, & c^2, & b^2 \\ 1, & c^2, & 0, & a^2 \\ 1, & b^2, & a^2, & 0 \end{vmatrix} = \begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 1, & 0, & p^2, & q^2, & r^2 \\ 1, & p^2, & 0, & c^2, & b^2 \\ 1, & q^2, & c^2, & 0, & a^2 \\ 1, & r^2, & b^2, & a^2, & 0 \end{vmatrix}$$

Similarly, from the known relation between the arcs joining four points on a sphere (Salmon's *Geometry of Three Dimensions*, p. 33),

$$\begin{vmatrix} 1, & \cos ab, & \cos ac, & \cos ad \\ \cos ba, & 1, & \cos bc, & \cos bd \\ \cos ca, & \cos cb, & 1, & \cos cd \\ \cos da, & \cos db, & \cos dc, & 1 \end{vmatrix} = 0.$$

If d be the centre of the small circle radius R, p and R are the sides of a right-angled spherical triangle of which ab is the hypotenuse, and $\cos ab = \cos p \cos R$, and so therefore

$$\begin{vmatrix} 1, & \cos p \cos R, & \cos q \cos R, & \cos r \cos R \\ \cos p \cos R, & 1, & \cos c, & \cos b \\ \cos q \cos R, & \cos c, & 1, & \cos a \\ \cos r \cos R, & \cos b, & \cos a, & 1 \end{vmatrix} = 0;$$

$$\therefore \sec^2 R \begin{vmatrix} 1, & \cos c, & \cos b \\ \cos c, & 1, & \cos a \\ \cos b, & \cos a, & 1 \end{vmatrix} + \begin{vmatrix} 0, & \cos p, & \cos q, & \cos r \\ \cos p, & 1, & \cos c, & \cos b \\ \cos q, & \cos c, & 1, & \cos a \\ \cos r, & \cos b, & \cos a, & 1 \end{vmatrix} = 0.$$

10352. (Professor CURTIS.)—The conics

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \quad a'x^2 + 2h'xy + \ldots + c' = 0,$$

cut each other in four points; prove that ordinates of these points are given by $Py^4 + Qy^3 + Ry^2 + Sy + T = 0$, where

$P = 4(ah')(hb') - (ab')^2$,

$Q = 4\left[(ag')(hb') + (ah')\left\{(gb') + 2(hf')\right\} - (ab')(af')\right]$,

$R = 4\left[(ag')\left\{2(hf') + (gb')\right\} + (ah')\left\{(hc') + 2(gf')\right\} - (af')^2 - \frac{1}{2}(ab')(ac')\right]$,

$S = 4\left[(ah')(gc') + (ag')\left\{(hc') + 2(gf')\right\} - (af')(ac')\right]$,

$T = 4(ag')(gc') - (ac')^2$.

Solution by the PROPOSER.

$$ax^2 + 2(hy + g)x + (by^2 + 2fy + c) = 0,$$

$$a'x^2 + 2(h'y + g')x + (b'y^2 + 2f'y + c') = 0$$

are the conics; hence, eliminating x,

$$4\begin{vmatrix} a & hy+g \\ a' & h'y+g' \end{vmatrix}\begin{vmatrix} hy+g & by^2+2fy+c \\ h'y+g' & b'y^2+2f'y+c' \end{vmatrix} = \begin{vmatrix} a & by^2+2fy+c \\ a' & b'y^2+2f'y+c' \end{vmatrix}^2.$$

From this the coefficients of the powers of y may be collected.

10431. (W. J. GREENSTREET, M.A.)—A spherical raindrop, radius a, falls for t'', when its radius is found to be b. Supposing the increase of mass to be proportional to the surface, find its velocity at the end of the t''.

Solution by Professor ANDERSON, M.A.; C. BICKERDIKE; *and others.*

Neglecting the resistance of the air, the equation of motion is $(d/dt)(mv) = mg$, and the rate of increase of mass being proportional to the surface, dr/dt is constant. Hence, if T denote the whole time of falling, $r = (b-a)\,t/T + a$ at any time t. The above equation becomes

$$\left(\frac{b-a}{T}t + a\right)\left(g - \frac{dv}{dt}\right) = 3v\,\frac{b-a}{T},$$

whose solution is easily found to be

$$v = gT - \frac{\left[(b-a)\,t/T + a\right]^4 - a^4}{4(b-a)\left[(b-a)\,t/T + a\right]^3},$$

the velocity at the end of the time T being $\frac{1}{4}gT(1 + a/b)(1 + a^2/b^2)$.

3630. (Professor HUDSON, M.A.)—The centre of a spherical mass is describing a parabola freely about the focus; the sphere expands as it approaches the focus, its radius varying inversely as the square of the distance, so that it traces out a parabolic tube of variable bore. According to what power of the distance does the rate of tracing out volume vary?

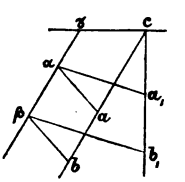
Solution by Professor ANDERSON, M.A.

The volume traced out by the sphere in a small element of time δt is $\pi r^2 . v . \delta t$, v being the velocity of the centre of the sphere and r its radius. But $r = a^3/\rho^2$, where a is a constant and ρ the focal distance of the centre. Also, $vp = h$, and $p^2 = m\rho$; therefore $\pi r^2 v \delta t = \pi h m^{-\frac{1}{2}} a^6 \rho^{-\frac{9}{2}} \delta t$. Hence the rate of describing volumes $\propto \rho^{-\frac{9}{2}}$.

2754. (SAMUEL ROBERTS, M.A.)—Show that, if (a, a_1), (b, b_1) are points of contact of tangents from two points on a cubic curve, and (a, b), (a_1, b_1) have the same connective, then the four points lie on a conic which passes through their tangentials.

Solution by Professor SCHOUTE.

When ab and a_1b_1 intersect in the point c of the cubic, and α, β, γ are the tangentials of $a\,(a_1)$, $b\,(b_1)$, c, the points α, β, γ are collinear as a, b, c, (a_1, b_1, c) are collinear. Now the lines abc, a_1b_1c, $\alpha\beta\gamma$ meet the cubic in nine points through which pass an infinite number of cubics; three of these nine (c, c, γ) are collinear, so the other six $(a, a_1, b, b_1, \alpha, \beta)$ lie on a conic.



[For another solution, see Vol. LI., p. 155.]

10461. (A. E. JOLLIFFE.)—A conic has a given self-conjugate triangle, and its director circle passes through a fixed point. Construct four fixed tangents to it. If a conic be drawn to circumscribe the triangle with the fixed point as focus, the corresponding directrix touches the given conic.

Solution by R. F. DAVIS, M.A.; PROMANATHA DATAR; *and others.*

Let ABC be a triangle; I, I_a, I_b, I_c its in- and ex-centres. Then ABC is self-conjugate with respect to any conic circumscribing the quadrilateral $II_aI_bI_c$. But, since I is the orthocentre of $I_aI_bI_c$, this conic must be a rectangular hyperbola.

Assuming the converse theorem, we have, by reciprocation, &c.

9719. (A. W. Panton, M.A.)—Prove that (1) there is in general one point, and only one, whose polar conic with regard to a given plane cubic is a circle, viz., the symmedian point of the triangle formed by the asymptotes; and (2) that the locus of points whose polar conics are rectangular hyperbolas is a right line, viz., the axis of perspective of the same triangle and its pedal triangle; also find (3) how the points are determined whose polar conics are—(i.) a pair of rectangular lines and (ii.) a pair of parallel lines; and (4) show that there are on the Hessian of the cubic three points forming a triangle, with reference to which any pair of corresponding points on the Hessian are isogonal conjugates.

Solution by Professor A. M. Nash, M.A.

(1) The equation of the cubic referred to the asymptotic triangle is

$$xyz = (ax+by+cz)^2(lx+my+nz), \text{ or } xyz = S^2L.$$

The polar conic of the point XYZ is

$$X(yz-2aSL-lS^2)+Y(zx-2bSL-mS^2)+Z(xy-2cSL-nS^2)=0,$$

which is similar to $Xyz+Yzx+Zxy=0$, since $S=0$ is the line at infinity; hence if the conic is a circle, XYZ is the symmedian point.

(2) If the conic is a rectangular hyperbola,

$$X\cos A+Y\cos B+Z\cos C=0.$$

(3) If the conic reduce to a pair of lines, XYZ must be on the Hessian; and (i.) if these lines are at right angles, the equation (2) must also be satisfied, so that the point must be one of the three intersections of that line and the Hessian; (ii.) if the lines are parallel, the equation $Xyz+\&c.=0$ must represent a parabola, so that the point must lie upon the ellipse which touches the sides of the triangle at their mid-points.

(4) The equation of any cubic can be put into the form

$$x^3+y^3+z^3=k(x+y+z)^3,$$

where the vertices of the triangle of reference are the points on the Hessian corresponding to three collinear inflexions. The Hessian is

$$xyz=kS(yz+zx+xy),$$

and it can easily be shown that the point corresponding to XYZ is YZ, ZX, XY.

10462. (W. J. Johnston, M.A.)—If a vector $\equiv ix+jy+kz$, as in quaternions, and we assume

$$i^2=j^2=k^2=ijk=+1 \quad (\text{so that } ij=ji=k, \&c.);$$

prove that (1) an *associative* and *commutative* algebra is thereby obtained; (2) on this system the scalar part of

$$a(b-c)+b(c-a)+c(a-b)=0,$$

gives the known theorem that, if A, B, C, D are four points in space,

$$\Sigma AB.CD\cos(\widehat{AB,CD})\equiv 0;$$

(3) interpret the vector part of the product of two vectors, also the vector part of the above identity; and (4) if $w \equiv i+j+k$, prove $(\omega+1)(\omega-3) = 0$, and interpret this.

Solution by Professors CURTIS, MUKHOPÀDHYÂY, *and others.*

(1) We have $(i^2)j = +1 \,.\, j = j$ and $i\,(ij) = ik = j$, therefore $(i^2)j = i\,(ij)$. Similarly, it is proved that $i\,(jk) = (ij)\,k$, &c., so that the associative principle holds.

The commutative principle also holds in this case; for

$$\alpha\beta - \beta\alpha = (xi+yj+zk)(x'i+y'j+z'k) = xx'+yy'+zz'+(yz'+y'z)\,i+\ldots.$$

(2) Hence $\quad S\alpha\beta = xx'+yy'+zz' = T\alpha \,.\, T\beta \,.\, \cos(\widehat{\alpha\,.\,\beta})$.

If DA, DB, DC be represented by a, b, c, then $(b-c)$, $(c-a)$, $(a-b)$ are BC, CA, and AB; and therefore, as $\Sigma S\alpha\,(b-c) = 0$, therefore

$$\Sigma AB \,.\, CD \cos(\widehat{AB, CD}) = 0.$$

(3) $\qquad V\alpha\beta = (yz'+y'z)\,i+(zx'+z'x)\,j+(xy'+x'y)\,k.$

This is proportional and perpendicular to an area of which the projections on the three principal planes are $yz'+y'z$, &c.

$$Va\,(b-c)+Vb\,(c-a)+Vc\,(a-b) = 0$$

signifies that these three vectors form a triangle.

(4) $\qquad \omega^2 = (i^2+j^2+k^2)+2\,(i+j+k) = 3+2\omega,$

therefore $\omega^2-2\omega-3 = 0$, therefore $(\omega+1)(\omega-3) = 0$.

That is, ω, as an operator, either reverses the direction of the vector operated on, or multiplies it by 3. Neither of these is possible with the operators i, j, k, therefore operators have in this system different qualities for different directions in space.

10479. (Professor GODEFROY.)—Si $x+y+z = 0$, on a

$$\left(\frac{y-z}{x}+\frac{z-x}{y}+\frac{x-y}{z}\right)\left(\frac{x}{y-z}+\frac{y}{z-x}+\frac{z}{x-y}\right) = 9.$$

Solution by Professor STEGGALL; Rev. H. SEWELL, M.A.; *and others.*

Writing $y-z = ax$, $z-x = by$, $x-y = cz$, we get

$$x+y+z = 0, \quad ax+by+cz = 0, \quad ax^2+by^2+cz^2 = 0;$$

hence, eliminating x, y, z, $\quad a\,(b-c)^2+b\,(c-a)^2+c\,(a-b)^2 = 0,$

or $\qquad \{a+b+c\}\,\{bc+ca+ab\} = 9abc,$

or $\qquad \{a+b+c\}\,\{1/a+1/b+1/c\} = 9,$

which is the result sought.

10508. (J. O'Byrne Croke, M.A.)—Taking the general equation of the parabola as $(ax+\beta y)^2+2gx+2fy+c=0$,

prove that (1) the focus and (2) the directrix are

$$x=\{\beta(g^2-f^2)+\beta c(\alpha^2+\beta^2)-2\alpha fg\}/\{2(\alpha f-\beta g)(\alpha^2+\beta^2)\} \quad \ldots (1);$$
$$y=\{\alpha(g^2-f^2)-\alpha c(\alpha^2+\beta^2)+2\beta fg\}/\{2(\alpha f-\beta g)(\alpha^2+\beta^2)\}$$
$$\beta x-\alpha y+\{(\alpha^2+\beta^2)c-f^2-g^2\}/\{2(\beta g-\alpha f)\}=0 \quad \ldots\ldots\ldots (2).$$

Solution by Rev. T. R. Terry, M.A.; Prof. Sarkar; *and others.*

The condition that the tangents from (x, y) to the conic should be at right angles is

$$(\alpha^2+\beta^2)(a^2x^2+2a\beta xy+\beta^2y^2+2gx+2fy+c)$$
$$-(a^2x+a\beta y+g)^2-(a\beta x+\beta^2y+f)^2=0,$$

or $$2(\beta g-\alpha f)(\beta x-\alpha y)+(\alpha^2+\beta^2)c-f^2-g^2=0,$$

which is consequently the equation to the directrix.

If (X, Y) be the focus, then, since the focus is the pole of the directrix,

$$\frac{a^2X+a\beta Y+g}{2\beta(\beta g-\alpha f)}=\frac{a\beta X+\beta^2Y+f}{-2a(\beta g-\alpha f)}=\frac{gX+fY+c}{(\alpha^2+\beta^2)c-f^2-g^2};$$

from the first two fractions we see that each $=\dfrac{1}{2(\alpha^2+\beta^2)}$,

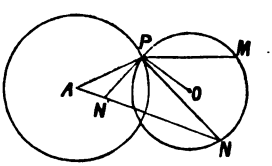
and, equating the first and third fractions to this, we get

$$X=[\beta(f^2-g^2)-\beta c(\alpha^2+\beta^2)+2\alpha fg]/\{2(\alpha^2+\beta^2)(\beta g-\alpha f)\},$$
$$Y=[\alpha(f^2-g^2)+\alpha c(\alpha^2+\beta^2)-2\beta fg]/\{2(\alpha^2+\beta^2)(\beta g-\alpha f)\}.$$

9952. (B. H. Steede, B.A.)—Describe a circle of a given coaxal system (of either species) cutting a given circle at a given angle.

Solution by the Proposer.

Let the given coaxal system have real points of intersection M, N. Let N′ be the inverse point of N with respect to the given circle whose centre is A. If a circle whose centre is O passing through M and N intersect the given circle at P at a given angle, the angle N′PM is given, for



$$N'PM=APO-APN'+OPM.$$

But $APN'=ANP$ and $OPM=90°-PNM$; therefore

$$N'PM=APO+90°-ANM=\text{a given angle.}$$

Therefore P is determined by the intersection of the given circle with a given segment of a circle.

The point determined by the remaining segment gives a circle of the coaxal system cutting the given circle at the supplemental angle. If the limiting points be real, we take the inverse point of one of these with respect to the given circle, and the solution is similar.

4105. (Professor HUDSON, M.A.) — A wheel in the form of a cylinder of radius R and thickness A has an axle of radius r and length a cut out of the same piece, the axes and centres of gravity being coincident. The whole is suspended with the axis horizontal by three vertical strings, one of which is coiled round the wheel and the other two round the axle at equal distances on either side of the wheel: prove that, if the first string be drawn up or let down in any way, the tensions of the other two will not be altered, provided $\frac{a}{A} = \frac{R^3(R-2r)}{r^3(2R-r)} + 1$.

Solution by Professor ANDERSON, M.A.

Let m denote the mass of the wheel alone, m' the mass of that part of the axle external to the wheel, T the tension of the string coiled round the wheel, and T′ the tension of the other strings. Also, let f denote the upward linear acceleration of the wheel, and $\dot{\omega}$ its angular acceleration. We have then the following equations:—

$$TR - 2T'r = \tfrac{1}{2}(mR^2 + m'r^2)\dot{\omega}, \quad T + 2T' - (m+m')g = (m+m')f, \quad f = r\dot{\omega}.$$

Eliminating T, we obtain, for the condition that T′ should be independent of f,

$$2R(m+m')f = (mR^2 + m'r^2)\dot{\omega},$$

or

$$2Rr(m+m') = mR^2 + m'r^2.$$

Hence

$$2Rr\left[R^2A + r^2(a-A)\right] = R^4A + r^4(a-A),$$

or

$$(a/A)(2r^3R - r^4) = 2r^3R - r^4 + R^4 - 2rR^3,$$

whence the stated result follows.

10100. (J. W. RUSSELL, M.A.)—It is commonly stated (without proof) in text-books that $x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$ is divergent if x is positive, and greater than unity; required a proof of this statement.

Note by the EDITOR.

The late Professor DE MORGAN, in his *Calculus*, p. 226, shows that the series is divergent when $x = 1{\cdot}3$, and states that the individual terms sooner or later increase without limit when x is anything greater than 1.

Moreover, in his *Summation of Divergent Series*, published in 1865, he proves that $1 + \frac{1}{2} + \frac{1}{3} + \ldots$ is divergent as follows:—Call the series S, then we have $S = (1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots) + (1 \ldots + \frac{1}{3} + \ldots)$, therefore $S = S + a$, where a is finite, therefore S is infinite. This proof, he adds, was given to him 37 years before by Professor SYLVESTER, who was then only 13 years old.

10190. (Professor MORLEY, B.A.)—Let a line cut a cubic at ABC, let the tangential of A be a, and let the contacts of tangents from A be O, P, Q, R. Then the following conics (denoting the repetition of a point by a power) pass through the point where Ra cuts the cubic

$$R^5,\ R^3BC,\ R^3O^2,\ R^3P^2,\ R^3Q^2,\ R^2OPQ,\ RB^2C^2,\ OPQAa,$$
$$O^3PQ,\ OP^3Q,\ OPQ^3,\ OPQBC.$$

Solution by Professor A. M. NASH, M.A.

(1) The tangents ARR, ARR form a conic through R, R, R, R; therefore these points are residual to AA, and therefore coresidual to a; therefore, &c.

(2) Similarly B, C, R, R, are coresidual to a, since ARR, ABC form a conic through them, and meeting the cubic again in AA, collinear with a.

(3) R, R, O, O are coresidual to a, since the lines ARR, AOO pass through them; and similarly for R, R, P, P; R, R, Q, Q.

(4) The polar conic of A touches the cubic at A and passes through R, O, P, Q; therefore these four points are coresidual to a.

(5) From the two lines ABC, ABC we see that B, B, C, C are coresidual to a.

(6) From the polar conic we see that O, P, Q, A are coresidual to R; therefore, &c.

(7) The polar conic and the tangent AOO form a cubic through the five given points O, O, O, P, Q, and meeting the cubic again in A, A, A, R; the tangents at A and R form a conic through these four, and meeting the cubic again in Ra, therefore these two points are coresidual to the five given points.

(8) The proof is the same as for (7), the line ABC being substituted for AOO.

3640. (The Rev. A. F. TORRY, M.A.)—The velocity of a particle freely describing a circle varies inversely as the ordinate to a fixed diameter: find the direction and law of the force.

Solution by Professor ANDERSON, M.A.

Let $vy = \gamma$ a constant, where v is the velocity of the particle, and y the ordinate. The component of the velocity parallel to the fixed diameter

is constant, and therefore the force is perpendicular to that diameter. Hence, resolving along the radius, we have

$$F\frac{y}{a} = m\frac{v^2}{a} = m\frac{\gamma^2}{ay^2} \text{ and } F = \frac{m\gamma^2}{y^3}.$$

10481. (Professor Cesáro.)—Soit $p_r = (1-x)(1-x^2)(1-x^3)\ldots(1-x^r)$. Démontrer l'identité $\frac{1}{p_n} - \frac{x}{p_1 p_{n-1}} + \frac{x^3}{p_2 p_{n-2}} - \ldots \pm \frac{x^{\frac{1}{2}n(n+1)}}{p_n} = 1.$

Solution by W. J. C. Sharp, M.A.; D. Edwardes, B.A.; *and others.*

In Salmon's *Higher Algebra*, p. 272, it is proved that

$$(1+xz)(1+x^2z)\ldots(1+x^nz)$$

$$= 1 + \frac{x(1-x^n)}{1-x}z + \frac{x^3(1-x^n)(1-x^{n-1})}{(1-x)(1-x^2)}z^2 + \ldots + \frac{x^{\frac{1}{2}n(n+1)}(1-x^n)\ldots(1-x)}{(1-x)\ldots(1-x^n)}z$$

$$= 1 + \frac{p_n x}{p_1 p_{n-1}}z + \frac{p_n x^3}{p_2 p_{n-1}}z^2 + \&c. + x^{\frac{1}{2}n(n+1)}z^n$$

$$= p_n\left\{\frac{1}{p_n} + \frac{x}{p_1 p_{n-1}}z + \frac{x^3}{p_2 p_{n-2}}z^2 + \ldots + \ldots + \frac{1}{p_n}x^{\frac{1}{2}n(n+1)}z^n\right\};$$

therefore if $z = -1$, $p_n = p_n\left\{\frac{1}{p_n} - \frac{x}{p_1 p_{n-1}} + \ldots + (-1)^n\frac{x^{\frac{1}{2}n(n+1)}}{p_n}\right\}$,

or $$\frac{1}{p_n} - \frac{x}{p_1 p_{n-1}} + \frac{x^3}{p_2 p_{n-2}} - \ldots + (-1)^n\frac{x^{\frac{1}{2}n(n+1)}}{p_n} = 1.$$

Similarly, putting $z = 1$,

$$\frac{(1+x)(1+x^2)\ldots(1+x^n)}{(1-x)(1-x^2)\ldots(1-x^n)} = \frac{1}{p_n} + \frac{x}{p_1 p_{n-1}} + \frac{x^3}{p_2 p_{n-2}} + \ldots + \frac{x^{\frac{1}{2}n(n+1)}}{p_n}.$$

In the same way, the identity

$$\frac{1}{p_n} = 1 + \frac{p_n}{p_1 p_{n-1}}x + \frac{p_{n+1}}{p_2 p_{n-1}}x^2 + \&c.$$

may be derived from the formula (Salmon, *loc. cit.*)

$$\frac{1}{(1-z)(1-xz)\ldots(1-x^{n-1}z)} = 1 + \frac{p_n}{p_1 p_{n-1}}z + \frac{p_{n+1}}{p_2 p_{n-1}}z^2 + \&c.$$

1054. (F. R. J. Hervey.)—A conic has its axes parallel to the asymptotes of a rectangular hyperbola, and cuts it in four points. Prove

that the mean of the intersections lies midway between the centres of the curves.

Solution by R. KNOWLES, B.A.; M. BRIERLEY; *and others.*

Let the equations to the hyperbola and conic, centres O, O′, be

$$xy = a^2, \quad x^2 + cy^2 + dx + ey + f = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1, 2);$$

then the coordinates of O′ are $-d/2$, $-e/2c$. Substituting from (1) in (2), we get $x^4 + dx^3 + \&c. = 0$, $cy^4 + ey^3 + \&c. = 0$, to determine the points of intersection; hence for the mean point $x = -d/4$, $y = -e/4c$, which is the mid-point of OO′.

10532. (Professor ORCHARD, M.A.)—Find the condition that we may solve, as a quadratic, $x^4 + 27x^3 + 9x^2 + 3x + e = 0$.

Solution by Rev. H. SEWELL, B.A.; R. KNOWLES, B.A.; *and others.*

$$x^2 + \frac{e}{x^2} + 27x + \frac{3}{x} + 9 = 0, \quad \therefore \quad x^2 + \frac{e}{x^2} + 27\left(x + \frac{1}{9x}\right) + 9 = 0;$$

and this may be solved as a quadratic if $\frac{e}{x^2} = \frac{1}{81x^2}$, or $e = \frac{1}{81}$.

[When $e = -\frac{1}{7^7}$, the equation may also be solved as a quadratic.]

10155. (Professor HERMITE.)—Prouver que, pour toutes les valeurs laquelle a positives de la variable x, l'expression $(x^a - x^{1-a})/(x-1)$, dans réelles et désigne une quantité comprise entre zéro et l'unité, peut être représentée par $(2a-1)\,\theta$, en supposant $0 < \theta < 1$.

Solution by Rev. J. L. KITCHIN; Prof. BEYENS; *and others.*

In $(x^a - x^{1-a})/x - 1$, put $x - 1 = y$; then

$$(1+y)^{1-a}\left(\frac{(1+y)^{2a-1} - 1}{y}\right)$$

$$= (1+y)^{1-a}(2a-1)\left\{1 + \frac{2a-2}{2!}\,y + \frac{(2a-2)(2a-3)}{3!}\,y^2 + \ldots\right\},$$

or

$$(1+y)^{1-a}(2a-1)\left\{1 - (1-a)\,y + \frac{(1-a)(3-2a)}{3}\,y^2 - \frac{(1-a)(3-2a)(4-2a)}{3\,.\,4}\,y^3 + \&c.\right\}.$$

Now the expression within the brackets is $< 1 - y + y^2 - a$; therefore $< (1+y)^{1-a}(2a-1)/1+y$, or than $\frac{2a-1}{(1+y)^a}$; therefore $(2a-1)\,\theta$, where θ lies between 0 and 1, represents the value of the expression.

10339. (L. Bénézech.)—On donne deux cercles O, O′ et un point P. On mène deux rayons parallèles OA, O′A′; sur PA, PA′ comme côtés on construit un parallélogramme. Pour quelle position des rayons parallèles la diagonale issue de P est-elle égale à la somme des rayons des cercles O, O′?

Solution by Rev. T. R. Terry, M.A.; C. Bickerdike; *and others.*

Take C the middle point of OO′ as origin, and CO′ as axis of x. Let a, α be the polar coordinates of P, and r and r' the radii. Then, if OA, O′A′ be inclined at an angle θ to the axis of x,

$$\{2a\cos\alpha-(r+r')\cos\theta\}^2+\{2a\sin\alpha-(r+r')\sin\theta\}^2=(r+r')^2,$$

or
$$a=(r+r')\cos(\alpha-\theta).$$

This gives the following construction:—From P draw a perpendicular to CP meeting in Q the circle whose centre is C and whose radius is $r+r'$. Then CQ is the required direction for the radii OA, O′A′.

10247. (W. J. Greenstreet, M.A.)—A line XY is divided in Z, so that ZY = 2XZ. Draw through X a line such that chords intercepted by both circles may be equal.

Solution by J. J. Barniville; Prof. Nilkantha Sarkar; *and others.*

The circles referred to are apparently those whose diameters are ZY and XZ. Let a line through X intersect the circumferences in A, B, C, so that XA = BC, and let XZ = a. Let O be the ⅓-point of ZY, and OD be perpendicular to XC. Then

$$AX^2=a^2-AZ^2, \text{ and } BC^2=4(OB^2-OD^2)=4(a^2-4AZ^2).$$

Therefore $a^2=5AZ^2$, and A is a given point.

10526. (Professor Mathews.)—Two plane convex quadrilaterals ABCD, A′B′C′D′ are equal in area, and each side of one is equal to the corresponding side of the other (AB = A′B′, &c.): prove that

$$AC\,.\,BD=A'C'\,.\,B'D'.$$

Solution by Profs. Anderson, Zerr, Seraminatha, Aiyar, *and others.*

If a, b, c, d, e, f be the lengths of the sides and diagonals of a convex quadrilateral, the area Q is given by (Johnson's *Trig.*, p. 164)

$$16Q^2=(2ef+b^2+d^2-a^2-c^2)(2ef+a^2+c^2-b^2-d^2)$$
$$=4e^2f^2-(b^2+d^2-a^2-c^2)^2.$$

Hence, if e', f' be the diagonals of the other quadrilateral, $ef=e'f'$.

APPENDIX.

UNSOLVED QUESTIONS.

4218. (J. O'Callaghan.)—OA, OB are any two semi-diameters of an ellipse; through their extremities A, B, draw the perpendiculars AP, BP, intersecting in P; trace the locus of P, and find its length and area.

4222. (The Rev. A. F. Torry, M.A.)—Two rods OA, OB are fixed in the same vertical plane, with the point O upwards, at the same angle α to the vertical. The ends of a rod AB of length $2a$ slide on them. Show that if the centre of gravity of AB be its middle point, and the radius of gyration about it be k, the time of a complete small oscillation is

$$2\pi\left(\frac{a^2\tan^2\alpha + k^2}{ag\cot\alpha}\right)^{\frac{1}{2}}.$$

4228. (A. Martin, LL.D.)—A coin, radius r, is placed at the bottom of an empty hemispherical bowl, radius R, and then the bowl is filled with water, when the coin is just visible to an eye looking over the edge. How much must the bowl be tipped from the eye to hide the coin?

4230. (Professor Cayley, F.R.S.)—Show that the apparent contour of a Steiner's surface $(2xyz + y^2z^2 + z^2x^2 + x^2y^2 = 0)$ as seen from an exterior point on a nodal line (say the axis of z), projected on the plane of the other two nodal lines, is an ellipse passing through the four points $x = \pm 1$, $y = \pm 1$; and find the surface-contour, or curve of contact of the cone and surface.

4238. (W. S. B. Woolhouse, F.R.A.S.)—The separate probabilities of three partially dependent events are given, as well as the probabilities of the joint occurrence of every two events, and also the joint occurrence of all three. In a proposed case, however, it is known that two of the events have failed, but it is unknown which; determine the probability of the failure also of the remaining event.

4241. (R. W. Genese, B.A.)—A building a feet square has a walk b feet wide round it. Two persons are on the walk; find the chance that they can see each other.

4243. (C. W. Merrifield, F.R.S.)—Find a rectangular parallelepiped, such that its edges, the diagonals of its faces, and the diagonals of the solid, shall all be integral. [The Proposer states that he has not been able to solve this problem, and would therefore be glad of a solution, or of a proof that it is impossible.]

4244. (Christine Ladd, B.Sc.)—Prove by geometry the construction of a regular pentagon given in *Crelle's Journal*, Bd. LXXV., S. 13.

4245. (A. Renshaw.)—Discuss the effect of variation of temperature on the figures of the large specula in Newtonian, Gregorian, and Cassegrainian reflecting telescopes.

4246. (T. T. Wilkinson, F.R.A.S.)—The straight line AB is a tangent to the circle whose centre is O, and XX′ is a diameter drawn through O parallel to AB; also, from any point T lines are drawn touching the circle at N. N′, respectively. Prove that, if NC, N′C, and TC be drawn cutting XX′ in n, n', and t, we shall have $nt = n't$.

4247. (W. S. Burnside, M.A.)—Show how to determine the number and positions of all the straight lines which can be drawn touching a curve of the former degree, so that the point of contact may be the middle point of the chord intercepted by the curve on each of the straight lines.

4248. (T. Cotterill, M.A.)—In the system of parabolas inscribed to a triangle, the directrix, the tangent at the vertex, the latus rectum, and the axis envelope tripods; find triangles by which the two last can be generated as tangents to the vertices of their inscribed parabolas. More generally, lines so invariably connected with the parabolas, that their equations, referred to the usual axes, are of the form $x \sin \alpha + y \cos \alpha = kp$ (α and k being constant and p the parameter) envelope tripods.

4249. (R. Moon, M.A.)—A horizontal cube closed at both ends is filled with air. At the time t every particle of the air is devoid of velocity; but the portion to the right of a vertical plane which bisects the tube has the uniform density D, while the air to the left of the same plane has the density 2D. Determine the pressure at the time t of the air in either half of the tube upon that in the other half.

4250. (N'Importe.)—If points P, Q, R be taken at random, one on each of the sides BC, CA, AB of the triangle ABC, find the probability of the area of the triangle PQR being greater than $\frac{1}{4}(1-n)$ of the triangle ABC, n being positive and < 1. (See the solutions to Question 2612, on p. 109 of Vol. x.)

4258. (Professor Evans, M.A.)—A person draws 10 cards from a full pack; find the chance that he has 30 spots.

4260. (Professor Malet, M.A.) — In connexion with Dr. Booth's Question 4201, prove the following theorem:—If a quadric be inscribed in a twisted quadrilateral, and the tangent planes through two consecutive sides be given, prove that the plane passing through the four points of contact, passes through a fixed point on one of the diagonals of the quadrilateral.

4261. (The Rev. J. L. Kitchin, M.A.)—A plank is resting against a vertical wall, with one end on a horizontal plane, the frictions of wall and plane being just sufficient to maintain equilibrium; and a sphere of given mass and radius is placed upon the plank at its upper end; find (1) the motion of the plank and sphere, and (2) the curve described by the centre of the sphere.

4262. (I. H. Turrell.) A square is divided at random by two straight cuts; find the probability that one of the pieces is a pentagon.

4269. (The Rev. Dr. Booth, F.R.S.)—If a, b, c, d, e, f, g be chords drawn from any point on the circumference of a circle to the seven angles of an inscribed regular heptagon; prove that

$(a+g)(b+f)(c+e) = d^3$.....(1), and that $a+c+e+g = b+d+f$.....(2).

4271. (The Editor.)—Let $a > b > c$ be the three semiaxes of an ellipsoid in the order of magnitude. Through the directrix of the principal section in the plane of a, b, let two planes be drawn parallel to the planes of the circular sections of the surface. These planes possess analogous properties to the directrix plane of a surface of revolution. For example, show that if from any point on the surface of an ellipsoid perpendiculars be drawn on these two planes, the rectangle contained by the perpendiculars will have a constant ratio to the square on the distance of this point from the corresponding focus of the principal section in the plane of a, b. When the ellipsoid becomes a surface of revolution round the major axis, the two directrix planes coalesce, the two perpendiculars become one, and we obtain the corresponding well-known property of surfaces of revolution and the conic sections.

4273. (Professor Townsend, M.A., F.R.S.) — Supposing the same plane curve to be at once the free orbit of a material particle and the free catenary of a uniform cord, under the action of corresponding central forces emanating from the same point in its plane, and depending only on the distance from that point; show that the radii of exhaustion, of the velocity of description in the former case, and of the tension of equilibrium in the latter case, are in general equal.

4277. (W. S. B. Woolhouse, F.R.A.S.)—Equal segments are cut off from two opposite corners of a given rectangle, the lines of section being parallel to the diagonal; determine their magnitude such that, if five points be taken at random on the remaining surface, the probability of their forming the apices of a convex pentagon shall be a maximum.

4278. (Professor Minchin, M.A.)—If any biquadratic

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4$$

be transformed by the substitutions $x^2 = uv$, $bx + c = u + v$, where b and c are arbitrary, and the biquadratic be then assumed to take the form $(u^2 + \theta u + \phi)(v^2 + \theta v + \phi)$, show that ϕ is determined by the equation

$$(a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3)\,\phi^3 - (a_0a_1a_4 - 3a_0a_2a_3 + 2a_1^2a_3)\,\phi^2 - (a_0a_3a_4 - 3a_1a_2a_4 + 2a_1a_3^2)\,\phi + (a_1a_4^2 - 3a_2a_3a_4 + 2a_3^3) = 0,$$

or $G\phi^3 - 3G_2\phi^2 + 3G_4\phi - G_6 = 0$, where $(GG_1G_2G_3G_4G_5G_6)(x, y)^6$ is the sextic covariant whose source is G.

(a) Show how all the roots of the biquadratic are found from any one root of the reducing cubic.

(b) Show that, if the roots of the biquadratic are $\alpha, \beta, \gamma, \delta$, the values of ϕ are given by the relation

$$\phi^2 = \frac{a_4}{x_0} + \frac{16}{a_0^3} \cdot \frac{a_0a_3^2 - a_1^2a_4}{(\alpha+\beta-\gamma-\delta)^2}.$$

4280. (Professor Genese, M.A.)—A stick has a uniform heavy string

suspended from its two ends. If the stick be made to turn round any point, find the locus of the vertex and the envelopes of the string and its directrix.

4281. (Matthew Collins, LL.D.) — A vertical circular tube contains three fluids whose densities are as 1, 2, s occupying spaces of 60°, 90°, 30°, respectively; find s so that the diameter of the semi-circular tube which contains the three fluids when at rest may be horizontal, or make a given angle with the horizon.

4285. (R. Tucker, M.A.)—On a table in a room in which are a number of flies are placed two lumps of sugar containing n and $2n$ particles, respectively: supposing the attraction to vary as the square of the number of constituent particles, and that each fly, after eating a particle, flies away, find the chance of the mth fly settling on the lump (n).

4288. (A. Renshaw.)—The 19th Sunday after Trinity fell this year (1873) on the 19th of October; when will this coincidence recur?

4295. (Professor Crofton, F.R.S.)—A force acting on a body containing two fixed points is balanced by their reactions; determine those reactions, on the supposition that their sum is to be the least possible; distinguishing the cases when the line of the forces passes between the fixed points, and outside them.

4296. (A. W. Panton, M.A.)—If the distances from any point on the axis of a Cartesian of the four points of intersection of the curve with its axis be given as the roots of the biquadratic

$$a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0;$$

form (1) a reducing cubic of this equation, whose roots shall be the distances of the three single foci from the triple focus; and (2) show that the expressions for these distances, in terms of the roots of the biquadratic, are

$$\frac{8G}{a_0^3(\alpha+\beta-\gamma-\delta)^2}, \quad \frac{8G}{8(\alpha+\gamma-\beta-\delta)^2}, \quad \frac{8G}{a_0^3(\alpha+\delta-\beta-\gamma)^2},$$

when $$G = a_0^2a_3 + 2a_1^3 - 3a_0a_1a_2.$$

4303. (T. Cotterill, M.A.)—1. In the system of eight points such that quadric surfaces through any seven pass through the eighth, prove that the line joining any two of the points is a line through two points of the twisted cubic through the other six. Hence, if six points are fixed, on every chord of the cubic through them there are pairs of points in involution, each pair forming, with the other six, eight such points.

2. If the quartic curve of intersection of two quadrics and a cubic intersect in six points, any plane cuts the curves in seven points on a conic. Explain particular cases.

4304. (The Rev. W. Roberts, M.A.)—Two equilateral hyperbolas, concentric with a given ellipse, and cutting each other orthogonally, touch the ellipse respectively at the points P, Q; show that the envelope of a third equilateral hyperbola, concentric with the former, and passing through the points P, Q, is an ellipse confocal with the given one.

4305. (J. Griffiths, M.A.)—If T denote a variable triangle inscribed in a given triangle ABC, so as to be homologous with it, H being the centre of homology, and α, β, γ, δ the centres of the four circles which touch the sides of T; prove that the seven nine-point circles of the different triangles that can be formed from H, α, β, γ, δ intersect in a point R, and that the locus of R is the nine-point circle of the triangle ABC.

4306. (A. Renshaw.)—Let O, Q be the centres of two fixed circles, of which the circumference of the latter passes through the centre of the former; and let any point P on the circumference of the circle (Q) be joined to two points M, N on the circle (O) such that PM and PN are each equal to the radius of the circle (O): then (1) prove that the chord MN passes through the fixed point Q; and (2) supposing P to revolve at an uniform rate in the circumference of the circle (Q), find the law of the revolution of the points M and N in the circumference of the circle (O); also investigate (3) whether by this process a uniform circular motion by P can or cannot be transformed into variable circular motion round Q, or *vice versâ ?*

4308. (W. Siverly.)—Let ABCD be a rectangle, AQ perpendicular to the plane AC at A; then if a variable triangle PRS, the plane of which is parallel to AQD, move with its angular points on BQ, BA, CD, respectively; to find the surface generated by PS. Is the solution given in Davies' "Hutton," Vol. II., page 545, correct?

4310. (Sir R. Ball, F.R.S.)—If α, β, γ be the roots of

$$ax^3 + 3bx^2 + 3cx + d = 0,$$

and if the angle θ be determined so that $\cos\theta = \dfrac{2b^3 - 3abc + a^2d}{2(b^2-ac)^{\frac{3}{2}}}$;

show that

$$\alpha - \beta = 2\sqrt{3}\,.\,\frac{(b^2-ac)^{\frac{1}{2}}}{a}\sin\frac{\theta}{3}, \quad \text{and} \quad 2\gamma - \beta - \alpha = 6\,\frac{(b^2-ac)^{\frac{1}{2}}}{a}\cos\frac{\theta}{3},$$

and from the former of these results deduce the equation of squared differences for the cubic, and from the latter the equation whose roots are $2\gamma - \beta - \alpha$.

4315. (S. Roberts, M.A.)—Show that a cubic can be drawn through the nine points of inflexion of a given cubic, and through the points at infinity determined by the asymptotes of the polar conic of the line at infinity and the polar line of the centre. (The centre of a cubic is the centre of the polar conic of the line at infinity.)

4319. (The Rev. J. Blissard, M.A.)—Let $\Sigma fx = f1 + f2 + \ldots + fx$, and let ${}_rC_n$ = the sum of the products of the n quantities $1\,.\,2\,\ldots\,n$ taken r together; then, if $\phi_1 n = \Sigma n \left(= \dfrac{n(n+1)}{1\,.\,2}\right)$,

$$\phi_2 n = \Sigma n \phi_1 n \left(= \frac{(3n-1)\,n\,(n+1)(n+2)}{1\,.\,2\,.\,3\,.\,4}\right), \quad \&c.$$

prove that (1) $\ldots\ldots \dfrac{\Delta^n 0^{n+r}}{1\,.\,2\,\ldots\,n} = \phi_r n$; (2) $\ldots\ldots {}_rC_n = \phi_r(n+1)$.

4337. (Sir R. Ball, F.R.S.)—If in any binary quantic $(a_0, a_1, \ldots a_n)(x, 1)^n$, or $F(x)$, x be changed into $\lambda + \frac{nF(\lambda)}{nx'F(\lambda) - F'(\lambda)}$, and the result be cleared of fractions by multiplying it by $\{nx'F(\lambda) - F'(\lambda)\}^n$; show that the coefficient of every power of x' in the expression thus obtained is a covariant of $F(\lambda)$.

4340. (Professor Clifford, F.R.S.)—1. If 1, 2, 3, 4 be four concyclic foci of an anallagmatic (bi-circular) quartic curve, on a plane or on a sphere, and P any point of the curve; the arc at P is equally inclined to the circles P12, P34.

2. The bitangent circles of an anallagmatic are arranged in four systems, orthotomic respectively of the four focal circles. Two bitangents of the same system can be drawn to cut orthogonally a given circle A. The four points of contact of these lie on a circle B_1 (*polar circle* of A *in that system*). There are polar circles B_2, B_3, B_4 in the other three systems, and the circles A, B_1, B_2, B_3, B_4 have a common radical axis.

3. The two bitangents of the same system through any given point are equally inclined to each of the two confocal anallagmatics which pass through that point.

4. The bitangent spheres of an anallagmatic quartic surface are arranged in five systems, orthotomic respectively of the five focal spheres. An infinite number of bitangents of the same system can be drawn to cut orthogonally a given sphere A; these envelope a cyclide, whose curve of contact with the anallagmatic surface lies on a sphere B, polar sphere of A in that system. The five polar spheres of A have with A a common radical plane.

5. The five tangent cyclides to an anallagmatic from any point have their focal spheres touched by the three confocal anallagmatics through that point.

6. A tangent cyclide from any focus is a *tore*.

DEF.—A *cyclide* is the envelope of a sphere touching three fixed spheres. When the centres of the three fixed spheres are in one straight line, the cyclide becomes a *tore*, or anchor-ring.

4341. (Professor Wolstenholme, M.A., Sc.D.)—A conic is drawn, osculating two given conics S, S′ in the points P, P′, respectively; the locus of the intersection of tangents at P, P′ is $\Delta' S^3 - \Delta S'^3 = 0$; and the envelope of PP′ is $(\Delta\Delta'^2)^{\frac{1}{3}} S + (\Delta^2\Delta')^{\frac{1}{3}} S' = F$, where Δ, Θ, Θ', Δ' are the fundamental invariants of S, S′, and F their harmonic locus so taken that its discriminant is $\Delta\Delta'(\Theta\Theta' - \Delta\Delta')$.

4346. (The Rev. W. Roberts, M.A.)—Let there be a surface, locus of points on confocal ellipsoids, at which the tangent planes cut off a constant volume from the principal planes; and let $U = \alpha$ represent a system of such surfaces, when the constant volume assumes different values. Also, let there be a surface, locus of points on the same confocal ellipsoids, at which the perpendicular drawn from the centre on the tangent plane is in a constant ratio to the perpendicular from the centre on the plane passing through the extremities of the axes; and let $V = \beta$ be the equation of the system, when the above-mentioned ratio is supposed to vary. Prove that the systems $U = \alpha$, $V = \beta$ cut each other orthogonally.

4347. (Professor Malet, F.R.S.)—If we put $\Delta(\theta) = \left\{(1-k^2\sin^2\theta)(1-k_1^2\sin^2\theta)(1-k_2^2\sin^2\theta)\left(1-\frac{k^2}{k_1^2}\sin^2\theta\right)\left(1-\frac{k^2}{k_2^2}\sin^2\theta\right)\right\}$,

$$L(\theta) = \int_0^\theta \frac{d\theta}{\Delta(\theta)}, \quad \text{and} \quad N(\theta) = \int_0^\theta \frac{\sin^4\theta}{\Delta(\theta)};$$

find an algebraic relation equivalent to the transcendental equation

$$L(\theta) + L(\phi) + L(\psi) = k^2\{N(\theta) + N(\phi) + N(\psi)\}.$$

4361. (W. B. Davis, B.A.) — Show that $r^4 - 2a^2r^2\cos 2t + a^4 = b^4$ represents the Ovals of Descartes, and that $r^2 = \frac{a^2\cos 2\theta}{\cos 2(t-\theta)}$ are equilateral hyperbolas having the same centre and one point in common with the Cartesian Oval.

4362. (Professor Burnside, M.A.)—Prove that confocal circular cubics cut at right angles: taking advantage of Prof. SYLVESTER's generation of these curves given in Quest. 1990. [*Reprint*, Vol. VI., p. 88.]

4363. (A. Renshaw.)—Supposing the sun and planets to have been originally a vast mass of matter in a gaseous state, and during the process of condensation to have broken up into rings which again ultimately became planets; is it possible to assign the law of condensation? That is to say, supposing, for example, Uranus to travel m miles and Saturn n miles per second in their orbits, is the superior velocity of the latter planet the resultant of the Uranian velocity, compounded with increasing velocity acquired by the revolving interior mass condensed until it arrives at the orbit of Saturn? And, further, can it be shown that the splitting up into rings, at certain periods or distances from the centre of revolution, was a necessary result of a certain ascertainable law of condensation?

4364. (The Rev. G. H. Hopkins, M.A.)—In what year before Christ did the sun enter each sign of the zodiac upon the first day of the month to which such sign corresponded?

4373. (Professor Malet, F.R.S.)—Solve the equations

$$(x^5/p_6)\,p_2 + yp_4 + zp_3 + wp_5 + p_6 = x(yp_2 + zp_1 + wp_3 + p_4) + (x^6/p^6),$$
$$(x^5/p_6)\,p_3 + yp_5 + zp_4 + wp_6 = x^2(yp_1 + z + wp_2 + p_3),$$
$$(x^5/p_6)\,p_4 + yp_6 + zp_5 = x^3(y + wp_1 + p_2), \quad (x^5/p_6)\,p_5 + zp_6 = x^4(p_1 + w),$$

being given the condition

$$p_1^3p_6^2 + p_1^2p_4^2p_5 - 2p_1^2p_2p_5p_6 - p_1^2p_3p_4p_6 + p_1p_2^2p_5^2 - 2p_1p_4p_5^2 + 3p_1p_3p_5p_6 + p_1p_2p_3^2p_6 - p_1p_2p_3p_4p_5 + p_5^3 - p_2p_3p_5^2 + p_3^2p_4p_5 - p_3^3p_6 = 0.$$

4381. (A. Martin, LL.D.)—A sphere, of radius r, *rolls* down the surface of another sphere, of radius R, placed on a horizontal plane. The surfaces of both spheres and plane are rough enough to secure perfect rolling. Find the point of separation of the spheres, and the path of the centre of the upper one.

4388. (Professor Crofton, F.R.S.)—1. A plane convex disc is kept in equilibrium by tangential stress of uniform intensity, changing sign at

four points A, B, C, D of the contour. Show that ABCD must be the vertices of a parallelogram; and that the areas of the four segments cut off from the disc must be such that

segment on AB + seg. on CD = seg. on BC + seg. on DA.

2. If the stress change sign at *six* points, then calling the sides of the hexagon so determined 1, 2, 3, 4, 5, 6, and the areas of the segments they cut off S_1, S_2, S_3, &c., the sides 1, 3, 5 must be parallel and equal to the sides of a triangle (whose area we call Δ_1); and also 2, 4, 6 parallel and equal to the sides of another triangle (area Δ_2); and we must besides have the condition $S_1+S_3+S_5+\Delta_1 = S_2+S_4+S_6+\Delta_2$.

4393. (The Rev. W. Roberts, M.A.)—Being given

$$x = \frac{(1+uv)(u-v)}{(1-uv)^2}, \quad y = \frac{2\left\{uv\,(1-u^2)(1-v^2)\right\}^{\frac{1}{2}}}{(1-uv)^2},$$

find what dx^2+dy^2 becomes in terms of u, v, du, dv, and explain accurately the geometrical meaning of the result.

4415. (Professor Crofton, F.R.S.)—In a Cartesian oval whose two internal foci coincide, prove that the difference of the two arcs intercepted by any two transversals from the external focus, is equal to a straight line which may be found. [The above curve is one species of *limaçon*; it is the inverse to an ellipse from a focus: the double point limaçon being inverse to an hyperbola.]

4416. (Colonel Clarke, C.B., F.R.S.)—An infinite number of points are distributed uniformly on the circumference of a circle, and lines are drawn through every pair; find the law of distribution or density of the points of intersection of these lines.

4433. (Lanivicensis.) — Prove by elementary Analytical Geometry that, if l, m, n are the sides of a triangle, and p, q, r the perpendiculars thereon from a variable point P, then $l/p+m/q+n/r = 0$ is the equation to the circle circumscribing the triangle.

4440. (Professor Wolstenholme, M.A., Sc.D.) — If $x_1, x_2 \ldots x_n$ be connected by the equation

$$x_1^2+x_2^2+\ldots+x_n^2-x_1x_2-x_2x_3+\ldots-x_{n-1}x_n = 1,$$

show that the maximum and minimum values of $x_1^2+x_2^2+\ldots+x_n^2$ will be the values of $\left(1+\cos\frac{r\pi}{n+1}\right)^{-1}$, where r has all integral values from 1 to n.

4442. (Rev. W. A. Whitworth, M.A.)—1. If n letters, $\alpha, \beta, \gamma, \delta, \ldots \mu, \nu$, be placed in a row at random, show that (1) the chance that there occur none of the n sequences, $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, $\ldots$ $\mu\nu$, $\nu\alpha$, is ϵ_{n-1}^{-1}; (2) the chance that there occur none of the $n-1$ sequences, $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, $\ldots$ $\mu\nu$, is $\epsilon_{n-1}^{-1}+n^{-1}\epsilon_{n-2}^{-1}$; and (3) that, if the n letters $\alpha, \beta, \gamma, \delta, \ldots \mu, \nu$ be arranged in *circular* order, the chance that there occur none of the n sequences, $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, $\ldots$ $\mu\nu$, $\nu\alpha$, is $\frac{1}{2}(1+\epsilon_n^{-2})$. [For notation, see Question 2637.]

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